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# New Proofs of Euclid's and Euler's Theorems

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In this note we give a new proof of the existence of infinitely many prime numbers. There are several different proofs with many variants, and some of them can be found in [1, 3, 4, 5, 6]. This proof is based on a simple counting argument using the inclusion-exclusion principle combined with an explicit formula. A different proof based on counting arguments is due to Thue (1897) and can be found in [6] together with several generalizations, and a remarkable variant of it was given by Chaitin [2] using algorithmic information theory. Moreover, we prove that the series of reciprocals of the primes diverges. Our proofs arise from a connection between the inclusion-exclusion principle and the infinite product of Euler.

Let  $\{p_i\}_i$  be the sequence of prime numbers, and let us define the following recurrence:

$$a_0 = 0, \quad a_{k+1} = a_k + \frac{1 - a_k}{p_{k+1}}.$$

Let us note that the  $N$ th term  $a_N$  generated by this recurrence coincides with

$$a_N = \sum_i \frac{1}{p_i} - \sum_{i < j} \frac{1}{p_i p_j} + \sum_{i < j < k} \frac{1}{p_i p_j p_k} - \dots + (-1)^{N+1} \frac{1}{p_1 \dots p_N},$$

and can be given in a closed form as

$$a_N = 1 - \prod_{i=1}^N \left(1 - \frac{1}{p_i}\right),$$

which implies that  $0 < a_N < 1$ , since each factor is strictly positive and less than one.

Now, we are ready to prove the classical Euclid's theorem:

**Theorem 1.** *There are infinitely many prime numbers.*

*Proof.* Let us suppose that  $p_1 < p_2 < \dots < p_N$  are all the primes. For any  $x \geq 1$ , and for  $i = 1, \dots, N$ , let  $A_i$  be the set of integers in  $[1, x]$  that are divisible by  $p_i$ . Then, the number of positive integers in  $[1, x]$  is obtained by applying the inclusion-exclusion formula to find the cardinality of  $\cup_{i=1}^N A_i$ :

$$[x] = 1 + \sum_i \left[ \frac{x}{p_i} \right] - \sum_{i < j} \left[ \frac{x}{p_i p_j} \right] + \sum_{i < j < k} \left[ \frac{x}{p_i p_j p_k} \right] - \dots + (-1)^{N+1} \left[ \frac{x}{p_1 \dots p_N} \right],$$

where  $[s]$  denotes the integral part of  $s$  as usual. Since

$$\lim_{x \rightarrow \infty} x^{-1} \left[ \frac{x}{t} \right] = \frac{1}{t},$$

we reach a contradiction,

$$1 > a_N = \sum_i \frac{1}{p_i} - \sum_{i < j} \frac{1}{p_i p_j} + \sum_{i < j < k} \frac{1}{p_i p_j p_k} - \dots + (-1)^{N+1} \frac{1}{p_1 \dots p_N} = 1,$$

and the proof is finished. ■

Let us observe from the previous proof that the asymptotic density  $D(p_1, \dots, p_N)$  of the set of integers divisible by none of  $p_1, \dots, p_N$  is exactly

$$D(p_1, \dots, p_N) = 1 - \sum_i \frac{1}{p_i} + \sum_{i < j} \frac{1}{p_i p_j} - \sum_{i < j < k} \frac{1}{p_i p_j p_k} + \dots + (-1)^N \frac{1}{p_1 \dots p_N},$$

that is,

$$1 - a_N = D(p_1, \dots, p_N) = \prod_{j=1}^N \left( 1 - \frac{1}{p_j} \right),$$

and let us define  $D = \lim_{N \rightarrow \infty} D(p_1, \dots, p_N)$ . Then, by taking logarithms, we obtain that

$$\sum_p \ln \left( 1 - \frac{1}{p} \right)$$

converges if  $D > 0$  and diverges if  $D = 0$ . Since  $\sum_p \frac{1}{p}$  converges if and only if  $\sum_p \ln \left( 1 - \frac{1}{p} \right)$  does, it is enough to show that  $D = 0$  to obtain:

**Theorem 2.** *The series  $\sum_p \frac{1}{p}$  diverges.*

*Proof.* Let us show that  $D > 0$  and the convergence of  $\sum_p \frac{1}{p}$  cannot hold simultaneously. To this end, let us take  $0 < \varepsilon < D$ , and choose  $N$  big enough so that

$$\varepsilon < D(p_1, \dots, p_N) \quad \text{and} \quad \sum_{p > p_N} \frac{1}{p} < \varepsilon.$$

Now the asymptotic density of the integers which are not divisible by any of the primes  $p_1, \dots, p_N$  is bounded below by  $\varepsilon$ . However, those integers must be divisible by some prime  $p > p_N$ , so their density is bounded above by

$$\sum_{p > p_N} \frac{1}{p} < \varepsilon,$$

a contradiction. Hence  $D = 0$  and the series diverges. ■

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## A Note on Covering a Square of Side Length $2 + \epsilon$ with Unit Squares

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In [1] Soifer posed the following problem: “Find the smallest number  $\Pi(n)$  of unit squares that can cover a square of side length  $n + \epsilon$  [for some  $\epsilon > 0$ ].” For small values of  $n$  the estimates presented in [1] are:  $5 \leq \Pi(2) \leq 7$  and  $10 \leq \Pi(3) \leq 14$ . The aim of this paper is to improve the lower bounds. We show that  $\Pi(2) \geq 6$  and  $\Pi(3) \geq 11$ , i.e., we show that it is impossible to cover a square of side length greater than 2 with five unit squares and it is impossible to cover a square of side length greater than 3 with ten unit squares.

**Lemma.** *Let  $S$  be a square of side length 1, let  $\delta > 0$ , and let  $l_1$  and  $l_2$  be straight lines parallel to each other with distance  $1 + \delta$ . Moreover, let both  $l_1$  and  $l_2$  have a nonempty intersection with  $S$ . Denote by  $s_i$  the length of  $l_i \cap S$  for  $i \in \{1, 2\}$ . Then  $s_1 + s_2 < 1$ .*

*Proof.*  $S$  has a nonempty intersection with  $l_1$  and  $l_2$ . Therefore no side of  $S$  is parallel to  $l_1$ . We can assume that  $l_1$  is on the left side of  $l_2$  as in Figure 1. Denote by  $a$  the vertex of  $S$  that lies on the right side of  $l_2$  (obviously, there is only one such vertex). Let  $p_1$  denote the distance between  $l_2$  and  $a$  and let  $w_1$  denote the length of the longest segment which is contained in  $S$  and is parallel to  $l_2$  and whose distance to  $a$  equals  $1 + \delta$ . It is easy to see that  $w_1 = s_1 + s_2$  (see Figure 1). Let  $w$  denote the length of the longest segment which is contained in  $S$  and is parallel to  $l_2$  and whose distance from  $a$  equals 1. From  $\delta > 0$  we deduce that  $w_1 < w$ . To find  $w$  observe that  $s_3 = \tan \frac{\alpha}{2}$  and  $1 - s_3 = w \cos \alpha$  in Figure 1. Consequently,

$$w = \left(1 - \tan \frac{\alpha}{2}\right) \frac{1}{\cos \alpha} = \left(1 - \tan \frac{\alpha}{2}\right) \frac{1 + \tan^2 \frac{\alpha}{2}}{1 - \tan^2 \frac{\alpha}{2}} = \frac{1 + \tan^2 \frac{\alpha}{2}}{1 + \tan \frac{\alpha}{2}}.$$

This value is smaller than 1, because  $\tan \frac{\alpha}{2} < 1$  for  $0^\circ < \alpha < 90^\circ$ . Thus  $s_1 + s_2 = w_1 < w < 1$ . ■