CS 624: Analysis of Algorithms

Fall 2024 Assignment 1

Due: September 21, 2024, on Gradescope

1. This question is based on Appendix C in CLRS, 4^th edition, question C.1-11 (page 1183). Argue that for any integers $n \geq 0, j \geq 0, k \geq 0$ and $j + k \leq n$:

   $\binom{n}{j+k} \leq \binom{n}{j} * \binom{n-j}{k}$

   Provide both an algebraic proof and an argument based on a method for choosing j + k items out of n. Give an example in which equality does not hold.

2. Decide whether each of the following statements is true or false, and prove that your conclusion is correct.

   (a) $2^{n+1} = O(2^n)$

   (b) $f(n) = O\big(g(n)\big)$ implies $2^{f(n)} = O\big(2^{g(n)}\big)$

3. Prove the correctness of the following algorithm for evaluating a polynomial

   $$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

   at a number x:

Algorithm 1 Horner(a,x)

```
p = a_n
for i = n − 1 to 0 do
    p := p · x + a_i
end for
return p
```

   This algorithm, as you probably know, is called *Horner's method*. You can use induction on the loop invariant using initiation, maintenance and termination.

4. Prove that if $f = O(g)$ and $g = O(h)$ then $f = O(h)$.

5. Give asymptotic tight bounds for T(n) for each of the recurrences. Justify your answers.

   (a) $T(n) = 2T(n/2) + n^3$

   (b) $T(n) = T(8n/11) + n$

   (c) $T(n) = 16T(n/4) + n^2$

   (d) $T(n) = 7T(n/2) + n^2 \log n$

   (e) $T(n) = 2T(n/4) + \sqrt{n}$

6. Problem 4.2 in Lecture notes 1 (page 7).

7. Problem 4.1 in Lecture notes 2 (page 13).

8. The Split3-Sort algorithm is defined as follows:

Algorithm 2 Split3-Sort(A,p,r)

```
1: if (A[p] > A[r]) then
2:    Swap A[p] with A[r]
3: end if
4: if (p + 1 < r) then
5:    k = ⌊(r − p + 1)/3⌋ // Round down
6:    Split3 − sort(A, p, r − k) // First two thirds
7:    Split3 − sort(A, p + k, r) // Last two thirds
8:    Split3 − sort(A, p, r − k) // First two thirds again
9: end if
```

   (a) Prove that the call to Split3-Sort(A, 1, n) correctly sorts the array $A[1..n]$ (**Hint:** I found the best way is to use induction, but be careful with the base case - notice line 4. What is the minimum difference between p and r?)

   (b) Write the recurrence formula for Split3-Sort and give the asymptotic bound on the run time (Θ notation).

   (c) Compare the run time from (b) to the run time of HeapSort, MergeSort and QuickSort. Is it better? Worse? Same?

9. Let $\{f_n : n = 0, 1, \dots\}$ be the Fibonacci sequence (where by convention $f_0 = 0$ and $f_1 = 1$).

   (a) This question is based on material from lecture notes 2. Show that $\sum_{n=1}^{\infty} \frac{nf_n}{2^{n-1}} = 20$. Do this by using a generating function as shown in the last section of the Lecture 2 notes, and differentiating. **Hint:** The derivative of $\frac{x}{1-x-x^2}$ is $\frac{1+x^2}{(1-x-x^2)^2}$.

   (b) Show why (in the same way as you proved the first part of this problem) you might think that $\sum_{n=1}^{\infty} nf_n = 2$. Then show why this could not possibly be true (it doesn't have to be a long answer, but it has to be convincing).