# Binary Search Trees

## CS 624 — Analysis of Algorithms

October 7, 2024

# Paths

For the next few slides, "graph" means "undirected graph".[1]

### Definitions (Path, Simple Path)

A **path** in a graph is a sequence $v_0, v_1, v_2, \ldots, v_n$ where each $v_j$ is a vertex in the graph and where each $v_i$ and $v_{i+1}$ are joined by an edge.

The **length** of the path $v_0, v_1, v_2, \ldots, v_n$ is $n$. A path can have length $0$.

A path in a graph is **simple** iff it contains no vertex more than once.

Usually we write $v_0 \to v_1 \to v_2 \to \ldots \to v_n$ to denote a path.

[1] See also Appendix B.4 (Graphs) and B.5 (Trees) in the textbook.

# Loops

## Definitions (Loop, Simple Loop)

A **loop**[2] is a path with at least one edge that begins and ends at the same vertex.

A loop $v_0 \to v_1 \to v_2 \to \ldots \to v_k$ is **simple** iff

1. $k \geq 3$ (that is, there are at least 4 vertices on the path), and
2. it contains no vertex more than once, except of course for the first and last vertices, $v_0 = v_k$, and
3. that (first and last) vertex occurs exactly twice

The definition of simple loop excludes trivial loops like $v_0 \to v_1 \to v_0$ and $v_0 \to v_1 \to v_2 \to v_1 \to v_0$.

[2] The textbook calls this a cycle.

# Trees (as Graphs)

**Definition (Tree)**

A **tree** is a undirected graph that contains no simple loops.

**Definition (Rooted Tree)**

A **rooted tree** is a tree with a distinguished vertex called the **root**.

# Ancestors and Descendants

### Definitions (Ancestor, Descendant, Parent, Child)

Let $T$ be a rooted tree with root $r$, and let $x$ and $y$ be vertices in $T$ (and either or both of them might be $r$).

- If there is a simple path from $r$ through $x$ to $y$, we say that $x$ is an **ancestor** of $y$ and $y$ is a **descendant** of $x$.
- Furthermore, if the part of the path from $x$ to $y$ consists of exactly one edge, we say that $x$ is the **parent** of $y$ and $y$ is a **child** of $x$.

Note that a vertex is both an ancestor and a descendant of itself.
But a vertex cannot be its own parent.

# Binary Trees

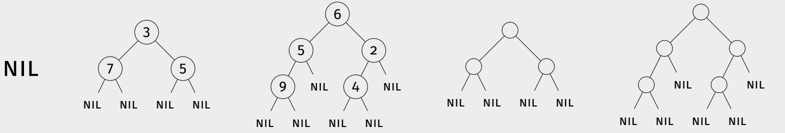
Recall from **Lecture 03**:

## Definition (Binary Tree)

A **binary tree** is either

- a **node** with two children, called *left* and *right*, which are also binary trees, and optionally a *data* field; or
- NIL, representing the empty tree

## Examples (Binary trees with and without data)



# Traversals

Reminder, tree traversal (any binary tree):

- **preorder traversal**
  1. visit the node itself *first*
  2. traverse the left child
  3. traverse the right child
- **inorder traversal**
  1. traverse the left child
  2. visit the node itself
  3. traverse the right child
- **postorder traversal**
  1. traverse the left child
  2. traverse the right child
  3. visit the node itself *last*

# Traversal Algorithms

**Algorithm 1** Preorder-Tree-Walk($x$)

```
if x ≠ NIL then
    visit(x)
    Preorder-Tree-Walk(left(x))
    Preorder-Tree-Walk(right(x))
end if
```

**Algorithm 2** Inorder-Tree-Walk($x$)

```
if x ≠ NIL then
    Inorder-Tree-Walk(left(x))
    visit(x)
    Inorder-Tree-Walk(right(x))
end if
```

**Algorithm 3** Postorder-Tree-Walk($x$)

```
if x ≠ NIL then
    Postorder-Tree-Walk(left(x))
    Postorder-Tree-Walk(right(x))
    visit(x)
end if
```

# Running Times of Traversal Algorithm

## Theorem

*If $x$ is the root of a binary tree with $n$ nodes, then each of the above traversals takes $\Theta(n)$ time.*

## Proof.

Let us define:

- $c$ = time for the test $x \neq nil$
- $v$ = time for the call to visit x
- $T(k)$ = time for the call to traverse a tree with k nodes

Then certainly we have

1. $T(0) = c$ and if the tree with $n$ nodes has a right child with $k$ nodes (so its left child must have $n - k - 1$ nodes), then
2. $T(n) = c + T(k) + T(n - k - 1) + v$

   We can show that $T(n) = (2c + v)n + c$.

□

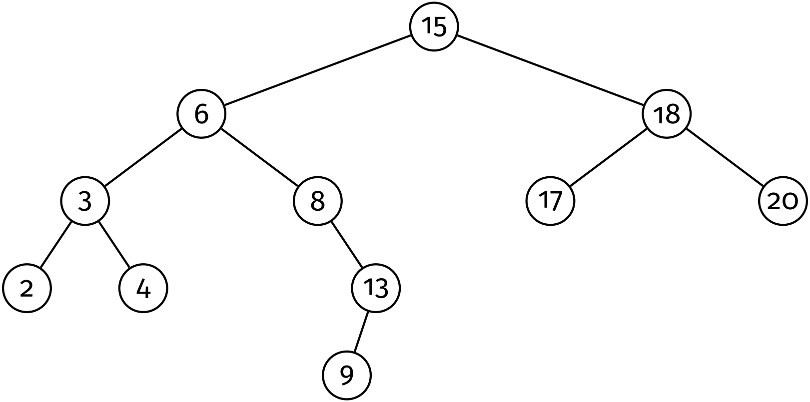
# Binary Search Trees (BSTs)

**Definition**

A **binary search tree** (**BST**) is a binary tree with data including a comparison key, where every node $x$ satisfies the BST properties:

1. If $y$ is a node on the left of $x$, $\mathrm{key}[y] \leq \mathrm{key}[x]$.
2. If $y$ is a node on the right of $x$, $\mathrm{key}[y] \geq \mathrm{key}[x]$.

# Example: BST



# Search

## Recursive version

**Algorithm 4** $\mathrm{TreeSearch}(x, k)$

**if** $x =$ NIL **or** $k = \mathrm{key}[x]$ **then**
    **return** $x$
**end if**
**if** $k < \mathrm{key}[x]$ **then**
    **return** $\mathrm{TreeSearch}(\mathrm{left}[x], k)$
**else**
    **return** $\mathrm{TreeSearch}(\mathrm{right}[x], k)$
**end if**

## Iterative version

**Algorithm 5** $\mathrm{TreeSearch}(x, k)$

**while** $x \neq$ NIL **and** $k \neq \mathrm{key}[x]$ **do**
    **if** $k < \mathrm{key}[x]$ **then**
        $x \leftarrow \mathrm{left}[x]$
    **else**
        $x \leftarrow \mathrm{right}[x]$
    **end if**
**end while**
**return** $x$

The running time is $O(h)$, where $h$ is the height of the tree.

# Minimum and Maximum

**Algorithm 6** $\text{TreeMinimum}(x)$

```
while left[x] ≠ NIL do
    x ← left[x]
end while
return x
```

**Algorithm 7** $\text{TreeMaximum}(x)$

```
while right[x] ≠ NIL do
    x ← right[x]
end while
return x
```

The running time is $O(h)$, where $h$ is the height of the tree.

# Successor and Predecessor

**Algorithm 8** $\text{TreeSuccessor}(x)$

```
if right[x] ≠ NIL then
    return TreeMinimum(right[x])
end if
y ← parent[x]
while y ≠ NIL and x = right[y] do
    x ← y
    y ← parent[x]
end while
return y
```

- The running time of TreeSuccessor on a tree of height $h$ is again $O(h)$, since the algorithm consists on following a path from a node to its successor, and the maximum path length is $h$.
- What happens when we apply this procedure to the node in the figure above whose key is 20?

# Running Time of Tree Procedures

TreePredecessor runs in a similar fashion with a similar running time.

**Theorem**

*The dynamic-set operations Search, Minimum, Maximum, Successor, and Predecessor can be made to run in $O(h)$ time on a binary search tree of height $h$.*

# Insert

**Algorithm 9** $\text{TreeInsert}(T, z)$

```
1:  y ← NIL
2:  x ← Root[T]
3:  while x ≠ NIL do
4:      y ← x
5:      if key[z] < key[x] then
6:          x ← left[x]
7:      else
8:          x ← right[x]
9:      end if
10: end while
11: parent[z] ← y
12: if y == NIL   /* only if T was empty */   then
13:     Root[T] ← z
14: else if key[z] < key[y] then
15:     left[y] ← z
16: else
17:     right[y] ← z
18: end if
```

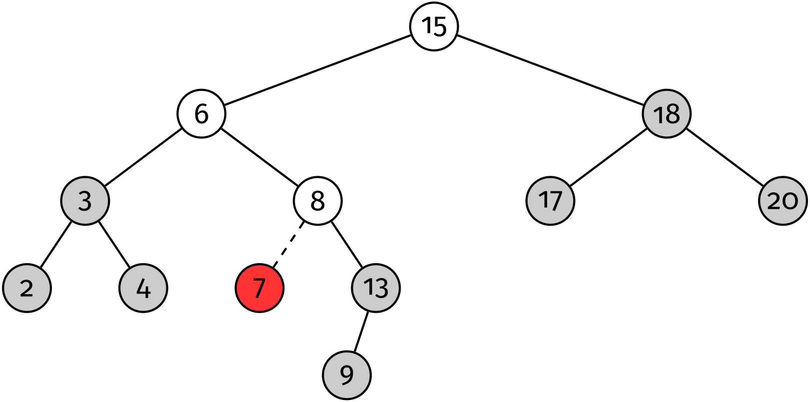
Lines 1–10:

- $x$ is the “lead explorer”, for where $z$ ought to be
- $y$ lags behind by one step
- when $x$ becomes NIL, $y$ is the last node on the path to $z$’s destination (if non-empty)

Lines 11–18:

- insert $z$ into the tree at “$x$”

# Insert Example




Obviously, Insert runs in $O(h)$ time on a tree of height $h$

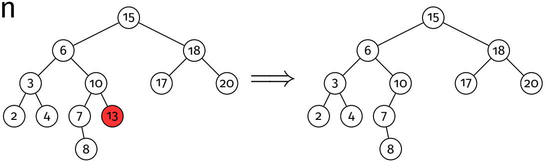
# Delete

- Deleting a node is more complicated.

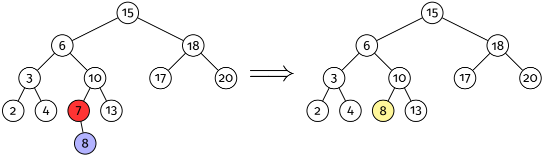
  If the node is buried within the tree, we will have to move some of the other nodes around.
- There are three cases to consider when deleting a node $d$:
  1. $d$ is a leaf
  2. $d$ has one child
  3. $d$ has two children

# Delete Examples

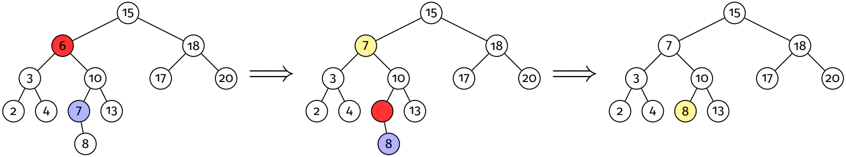
- Case I: 0 children




- Case II: 1 child




- Case III: 2 children



# Delete: Case by Case

- Case I - $d$ is a leaf. This case is trivial. Just delete the node. This amounts to figuring out which child it is of its parent, and making the corresponding child pointer nil.
- Case II: $d$ has one child. In this case, delete $d$ and "splice" its child to its parent – that is, make the parent's child pointer that formerly pointed to $d$ now point to $d$'s child, and make that child's parent pointer now point to $d$'s parent.
- Case III: $d$ has two children. In this case we can't simply move one of the children of $d$ into the position of $d$. What we need to do is find $d$'s successor and replace $d$ with it. Then delete $d$'s successor. Since the successor has at most one child (why?) then we revert to case I or II.

# Building a BST

An algorithm for building a binary search tree from an array $A[1 \mathrel{..} n]$:

**Algorithm 10** $\text{BuildBST}(A)$

```
1: T ← Create Empty Tree
2: for i ← 1 to n do
3:     TreeInsert(T, A[i])
4: end for
```

- What is the running time?
- Worst case: array already sorted, quadratic
- Best case: looks like $O(n \log n)$
- What does it remind us of?

# Modified Version of Partition

- $pivot \leftarrow A[p]$
- Let $L$ be the sequence of elements of $A[p+1\,..\,q]$ that are less than the pivot, in the order they appear in $A$
- Let $U$ be the sequence of elements of $A[p+1\,..\,q]$ that are greater than pivot, in the order they appear in $A$
- Rearrange the elements in $A[p\,..\,q]$ so that they appear like this:

$$L \; pivot \; U$$

- This may require more time than the original partition but not asymptotically more.

# Exercise

- Show that the comparisons needed to build a BST from an array $A[1 \mathinner{..} n]$ are exactly the same comparisons needed to do quicksort on the array, using ModifiedPartition.
- **Hint:** The comparisons in quicksort are against the pivot elements and the successive pivot elements are the successive elements added to the BST.

# Running Time for Constructing a BST

- We know that the average time for quicksort is $\Theta(nlogn)$.
- What is the "average time" for building a BST?
- It is the average over all possible permutations of the input array.
- This is exactly what we get with randomized quicksort.

**Theorem**

*The average time for constructing a BST is* $\Theta(nlogn)$.

# Running Time for Searching a BST

- The average search time in a BST is h, the height of a tree.
- What is the average height of a BST?
- We know the search time is the depth of a node.
- Which is the number of comparisons we make when inserting the node into the tree.

# Running Time for Searching a BST

- We see that the total expected number of comparisons is $O(n \log n)$.
- So the average number of comparisons is $O(\log n)$ per node.
- The average cost for search in a randomly build BST is therefore $O(\log n)$.
- There may be longer paths — in a linear tree the average search time is $O(n)$.
- However, the average height of a randomly build BST is $O(\log n)$.

# Running Time for Searching a BST

- Let $X_n$ be a random variable whose value is the height of a binary search tree on $n$ keys
- Let $P_n$ be the set of all permutations of those $n$ keys. (So the number of elements of $P_n$ is $n!$)
- Let $\pi$ to denote a permutation in $P_n$. $X_n$ is actually a function on $P_n$.
- Its value $X_n(\pi)$ when applied to a permutation $\pi \in P_n$ is the height of the binary search tree built from that permutation $\pi$
- We want to find $E(X_n)$, the *expectation* of $X_n$.
- This is by definition $\sum_{\pi \in P_n} p(\pi) X_n(\pi)$ where $p(\pi)$ denotes the probability of the permutation $\pi$.
- Assuming that all permutations have equal probability, $p(\pi) = \frac{1}{n!}$ for all $\pi$, and so $E(X_n) = \frac{1}{n!} \sum_{\pi \in P_n} X_n(\pi)$

# Note on Distribution

If $A$ and $B$ are two random variables on the same space $P_n$, then

$$\begin{aligned} E(A+B) &= \sum_{\pi \in P_n} p(\pi)\big(A(\pi)+B(\pi)\big) \\ &= \sum_{\pi \in P_n} p(\pi)A(\pi) + \sum_{\pi \in P_n} p(\pi)B(\pi) \\ &= E(A)+E(B) \end{aligned}$$

Note that $\max\{A,B\}$ is also a random variable on $P_n$ – its value at $\pi$ is just $\max\big\{A(\pi),B(\pi)\big\}$. And we have the useful inequality

$$\begin{aligned} E\big(\max\{A,B\}\big) &= \sum_{\pi \in P_n} p(\pi)\max\big\{A(\pi),B(\pi)\big\} \\ &\le \sum_{\pi \in P_n} p(\pi)\big(A(\pi)+B(\pi)\big) \\ &= E(A+B) = E(A)+E(B) \end{aligned}$$

# Expected Height of a BST

- Consider a permutation $\pi$. The root of the tree will be the first element of $\pi$.
- Suppose the root has position $k$ in the sorted list of keys.
- That means that there will be $k-1$ keys less than it and $n-k$ keys greater than it.
- So the left subtree will have $k-1$ elements and the right subtree will have $n-k$ elements.
- Those elements are also chosen randomly from sets of size $k-1$ and $n-k$ respectively, so we have

$$X_n(\pi) = 1 + \max\{X_{k-1}(\pi), X_{n-k}(\pi)\}$$

- This is our fundamental recursion.

# Expected Height of a BST

- Since each value of $k$ is chosen with the same probability (that probability being $\frac{1}{n}$), we have

$$E(X_n) = \sum_{k=1}^{n} \frac{1}{n} E\big(1 + \max\{X_{k-1}, X_{n-k}\}\big)$$

- An effective way to estimate it would be to set $Y_n = 2^{X_n}$.
- So $Y_n$ is itself a random variable defined on the set $P$ whose value on the permutation $\pi$ is $Y_n(\pi) = 2^{X_n(\pi)}$
- At first there is no intuitive significance to the reason for doing this. It's just that we can do better with the mathematics that way.
- Compute $E(Y_n)$ and use this to get a bound on $E(X_n)$.
- This step is also somewhat tricky if you haven't seen it before, but it is a general technique.

# Expected Height of a BST

$$\begin{aligned} Y_n(\pi) = 2^{X_n(\pi)} &= 2^{1+\max\{X_{k-1}(\pi), X_{n-k}(\pi)\}} \\ &= 2 \cdot 2^{\max\{X_{k-1}(\pi), X_{n-k}(\pi)\}} = 2 \cdot \max\{2^{X_{k-1}(\pi)}, 2^{X_{n-k}(\pi)}\} \\ &= 2 \cdot \max\{Y_{k-1}(\pi), Y_{n-k}(\pi)\} \end{aligned}$$

Since each value of $k$ is chosen with probability $\frac{1}{n}$:

$$\begin{aligned} E(Y_n) = \sum_{k=1}^{n} \frac{1}{n} \cdot 2E\big(\max\{Y_{k-1}, Y_{n-k}\}\big) &= \frac{2}{n} \sum_{k=1}^{n} E\big(\max\{Y_{k-1}, Y_{n-k}\}\big) \\ &\leq \frac{2}{n} \sum_{k=1}^{n} \big(E(Y_{k-1}) + E(Y_{n-k})\big) \end{aligned}$$

Each term is counted twice so we can simplify to get this:

$$E(Y_n) \leq \frac{4}{n} \sum_{k=1}^{n} E(Y_{k-1}) = \frac{4}{n} \sum_{k=0}^{n-1} E(Y_k)$$

# Expected Height of a BST

It is more convenient to use a strict equality, rather than an inequality. It turns out that we can assume this to be the case since we're really only concerned with an upper bound.

**Lemma**

*If $f$ and $g$ are two functions such that*

$$f(0) = g(0) \tag{1}$$

$$f(n) \leq \frac{4}{n} \sum_{k=0}^{n-1} f(k) \tag{2}$$

$$g(n) = \frac{4}{n} \sum_{k=0}^{n-1} g(k) \tag{3}$$

*then $f(k) \leq g(k)$ for all $k \geq 1$.*

# Expected Height of a BST

**Proof.**

We'll prove this by induction. The inductive hypothesis is that $f(k) \leq g(k)$ for all $k < n$. We know that this statement is true for $n = 1$ by the equation above. The inductive step is then to show that this statement remains true for $k = n$. To show this, we just compute as follows:

$$\begin{aligned} f(n) &\leq \frac{4}{n} \sum_{k=0}^{n-1} f(k) \\ &\leq \frac{4}{n} \sum_{k=0}^{n-1} g(k) = g(n) \end{aligned}$$

□

# Expected Height of a BST

- Based on this, we can assume that $E(Y_n) = \frac{4}{n}\sum_{k=0}^{n-1} E(Y_k)$.
- because any upper bound we obtain for $E(Y_n)$ from this identity will also be an upper bound for the “real” $E(Y_n)$.
- This is a similar trick to the one we used in deriving the average case running time of Quicksort.
- We can do something very similar here, although it is a little more complicated:
- $E(Y_{n+1}) = \frac{4}{n+1}\sum_{k=0}^{n} E(Y_k)$ and $E(Y_n) = \frac{4}{n}\sum_{k=0}^{n-1} E(Y_k)$
- We get rid of the denominators:
- $(n+1)E(Y_{n+1}) = 4\sum_{k=0}^{n} E(Y_k)$
- $nE(Y_n) = 4\sum_{k=0}^{n-1} E(Y_k)$

# Expected Height of a BST

- Now let us subtract and get: $(n+1)E(Y_{n+1}) - nE(Y_n) = 4E(Y_n)$
- $(n+1)E(Y_{n+1}) = (n+4)E(Y_n)$
- Divide both sides by $(n+1)(n+4)$. We get $\frac{E(Y_{n+1})}{n+4} = \frac{E(Y_n)}{n+1}$.
- If you look at it closely for a little while, you will see that if we now divide each side by $(n+2)(n+3)$, we will get something nice: $\frac{E(Y_{n+1})}{(n+4)(n+3)(n+2)} = \frac{E(Y_n)}{(n+3)(n+2)(n+1)}$.

# Expected Height of a BST

- And so if we define $g(n) = \frac{E(Y_n)}{(n+3)(n+2)(n+1)}$
- Then we have just derived the fact that g(n+1) = g(n)
- In other words, $g(n)$ is some constant. Call it $c$.
- Then we have $E(Y_n) = c(n+3)(n+2)(n+1) = O(n^3)$
- We are not done yet! We have to find $E(X_n)$

# Expected Height of a BST

- We know that there is a constant $C > 0$ and a number $n_0 \geq 0$ such that for all $n \geq n_0$, $E(Y_n) \leq Cn^3$. Hence for all $n \geq n_0$, $2^{E(X_n)} \leq E(2^{X_n}) = E(Y_n) \leq Cn^3$
- Taking the logarithm of both sides we get $E(X_n) \leq \log_2 C + 3 \log_2 n = O(\log n)$
- In other words, the expected height of a randomly build binary search tree is $O(\log n)$.