Depth-First Search

CS 624 — Analysis of Algorithms

November 20, 2024

Depth-First Search (DFS)

- DFS scans a graph (directed or undirected) $G = (V, E)$.
  Unlike BFS, no source vertex given!
- Adds 2 "**timestamps**" to each vertex, integers in $[1 .. 2|V|]$.
  - $d[v]$ = discovery time (v turns from White to Gray)
  - $f[v]$ = finishing time (v turns from Gray to Black)
- $\pi[v]$ = predecessor of v; a vertex u such that v was discovered during the scan of u's adjacency list
- Uses the same coloring scheme for vertices as BFS.
- The key mechanisms in this algorithm are the timestamps and a stack (implicit in the structure of recursive calls).

The DFS Algorithm

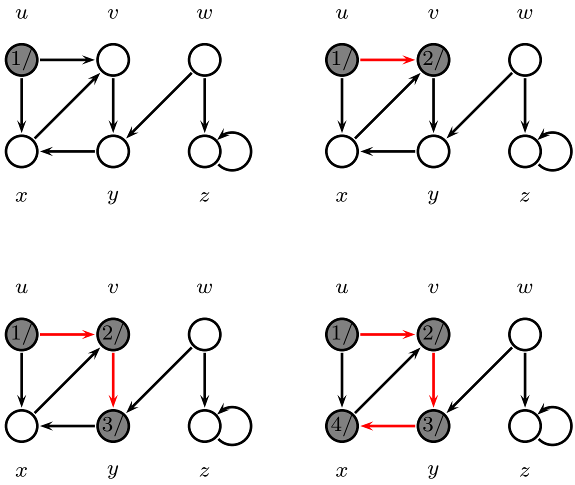
Algorithm 1 $\mathrm{DFS}(G)$

for each $u \in V[G]$ **do**
  $color[u] \leftarrow$ White
  $\pi[u] \leftarrow$ NIL
end for
$time \leftarrow 0$
for each $u \in V[G]$ **do**
  if $color[u] =$ White **then**
    $\mathrm{DFS\text{-}Visit}(u)$
  end if
end for

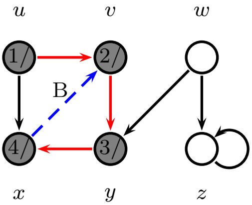
Algorithm 2 $\mathrm{DFS\text{-}Visit}(u)$

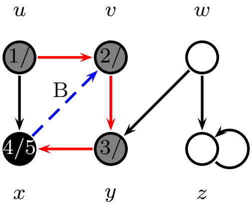
$color[u] \leftarrow$ Gray
$time \leftarrow time + 1$
$d[u] \leftarrow time$
for each $v \in Adj[u]$ **do**
  if $color[v] =$ White **then**
    $\pi[v] \leftarrow u$
    $\mathrm{DFS\text{-}Visit}(v)$
  end if
end for
$color[u] \leftarrow$ Black
$time \leftarrow time + 1$
$f[u].time \leftarrow time + 1$

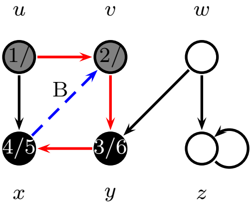
The DFS Algorithm – Example

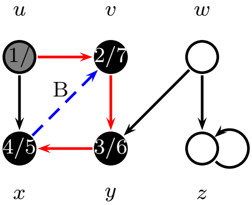


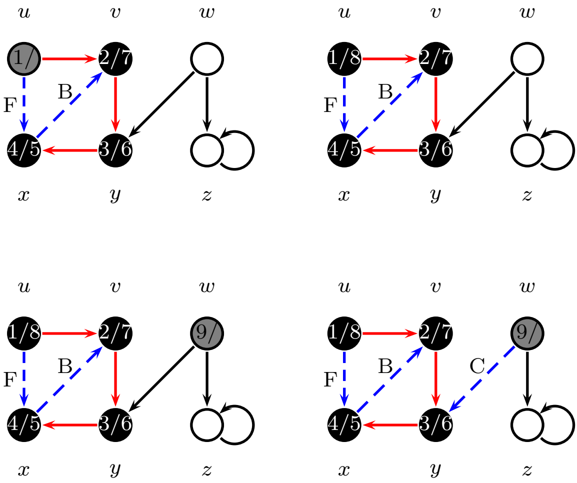
The DFS Algorithm – Example










u v w
1/ 2/7
F B
4/5 3/6
x y z
u v w
1/8 2/7
F B
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1/8 2/7 9/
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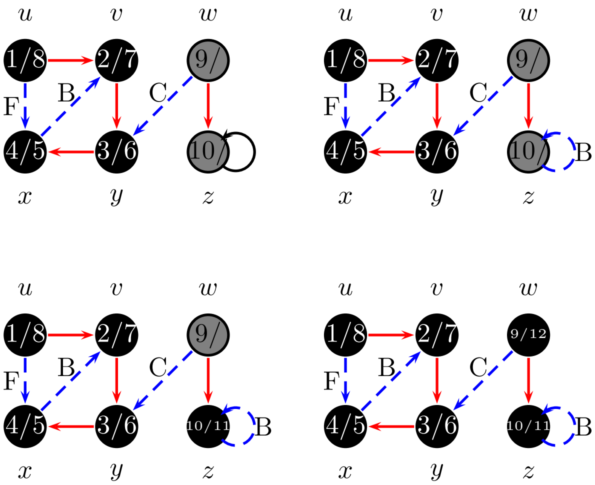
x y z

The DFS Algorithm – Example



The DFS Algorithm – Runtime and Properties

- The loops on lines 1-2 & 5-7 take $\Theta(V)$ time, excluding time to execute DFS-Visit.
- DFS-Visit is called once for each white vertex $v \in$ V when it's painted gray the first time.
- Lines 3-6 of DFS-Visit is executed |Adj[v]| times. The total cost of executing DFS-Visit is $\sum_{v\in V|Adj[v]|} = \Theta(E)$.
- Total running time of DFS is $\Theta(V + E)$.

The Parenthesis Theorem

Theorem (Parenthesis Theorem, Alternative Version)

Let $G = (V, E)$ and let $u, v \in V$ and suppose $d[u] < d[v]$ after DFS.

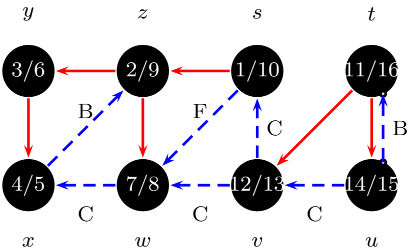
Then exactly one of the following cases holds:

1. $d[u] < f[u] < d[v] < f[v]$, *and neither u nor v is a descendant of the other*
2. $d[u] < d[v] < f[v] < f[u]$, *and v is a descendant of u*

So $d[u] < d[v] < f[u] < f[v]$ cannot happen.

- OK: ( ) [ ] ([ ]) [( )]
- Not OK: ([ )] [( ])

The Parenthesis Theorem – Example



(s (z (y (x x) y) (w w) z) s) (t (v v) (u u) t)

The Parenthesis Theorem

Theorem

For all u, v, exactly one of the following holds:

1. $d[u] < f[u] < d[v] < f[v]$ *or* $d[v] < f[v] < d[u] < f[u]$ *and neither* u *nor* v *is a descendant of the other.*
2. $d[u] < d[v] < f[v] < f[u]$ *and* v *is a descendant of* u.
3. $d[v] < d[u] < f[u] < f[v]$ *and* u *is a descendant of* v.

Corollary

v *is a proper descendant of* u *iff* $d[u] < d[v] < f[v] < f[u]$.

The Parenthesis Theorem

Proof.

- If $start[x] < start[y] < finish[x]$ then x is on the stack when y is first reached.
- Therefore the processing of y starts while x is on the stack, and so it also must finish while x is on the stack:
- we have $start[x] < start[y] < finish[y] < finish[x]$.
- The case when $start[y] < start[x] < finish[y]$ is handled in the same way.

$\square$

- Another way to state the parenthesis nesting property is that given any two nodes x and y, the intervals [start[x], finish[x]] and [start[y], finish[y]] must be either nested or disjoint.

Depth First Trees

- Predecessor subgraph defined slightly different from that of BFS.
- The predecessor subgraph of DFS is $G_\pi = (V, E_\pi)$ where $E_\pi = \{(\pi[v], v) : v \in V \text{ and } \pi[v] \neq NIL\}$.
- How does it differ from that of BFS?
- The predecessor subgraph G_π forms a depth-first forest composed of several depth-first trees.
- The edges in E_π are called tree edges.

Definition (Forest)

An acyclic graph G that may be disconnected.

White Path Theorem

Theorem

v is a tree descendant of u if and only if at time $d[u]$, there is a path $u \rightsquigarrow v$ consisting of only white vertices (except for u, which was just colored gray).

Proof.

One direction: (if v is a tree descendant of u then there is a white path $u \rightsquigarrow v$ at time $d[u]$) is obvious from the definition of a tree descendant (see the parenthesis theorem). $\square$

White Path Theorem

Cont. – Reverse Direction.

- Is it possible that v is not a descendant of u in the DFS forest?
- By induction on all the vertices along the path: Of course u is a descendant of itself.
- Let us pick any vertex p on the path other than the first vertex u, and let q be the previous vertex on the path (so it can be that $q = u$).
- We assume that all vertices along the path from u to q inclusive are descendants of u (inductive hypothesis).
- We will argue that p is also a descendant of u.

$\square$

White Path Theorem

Cont. – Reverse Direction.

- At time $d[u]$ vertex p is white [by assumption about the white path], so $d[u] < d[p]$.
- But there is an edge from q to p, so q must explore this edge before finishing.
- At the time when the edge is explored, p can be:
- **WHITE**, then p becomes a descendant of q, and so of u.
- **BLACK**, then $f[p] < f[q]$ [because $f[p]$ must have already been assigned by that time, while $f[q]$ will get assigned later].
- But since q is a descendant of u [not necessarily proper], $f[q] \leq f[u]$, we have $d[u] < d[p] < f[p] < f[q] \leq f[u]$, and we can use the Parenthesis Theorem to conclude that p is a descendant of u.

□

White Path Theorem

Cont. – Reverse Direction.

- **GRAY**, then p is already discovered, while q is not yet finished, so $d[p] < f[q]$.
- Since q is a descendant of u [not necessarily proper], by the Parenthesis Theorem, $f[q] \leq f[u]$.
- Hence $d[u] < d[p] < f[q] \leq f[u]$. So $d[p]$ belongs to the set $\{d[u], \ldots, f[u]\}$, and so we can use the the Parenthesis Theorem again to conclude that p must be a descendant of u.
- The conclusion thus far is that p is a descendant of u. Now, as long as there is a vertex on the remainder of the path from p to v, we can repeatedly apply the inductive argument, and finally conclude that the vertex v is a descendant of u, too.

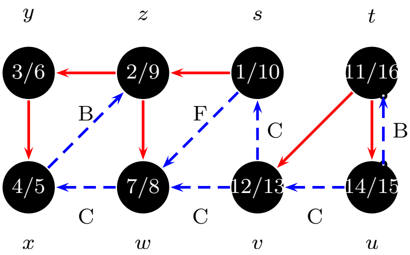
$\square$

Classification of Edges

- **Tree edge:** in the depth-first forest. Found by exploring (u, v).
- **Back edge:** (u, v), where u is a descendant of v (in the depth-first tree).
- **Forward edge:** (u, v), where v is a descendant of u, but (u, v) is not a tree edge.
- **Cross edge:** any other edge. Can go between vertices in same depth-first tree or in different depth-first trees.

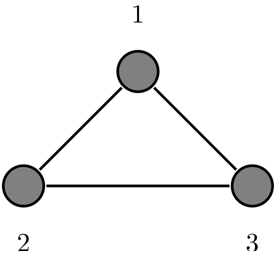
Edge type for edge (u, v) can be identified when it is first explored by DFS based on the color of v.

White $\rightarrow$ tree edge. Gray $\rightarrow$ back edge.
Black $\rightarrow$ forward or cross edge.



The edge $x \to z$ is discovered when exploring z, so it is a back edge.

Classification of Edges



Theorem

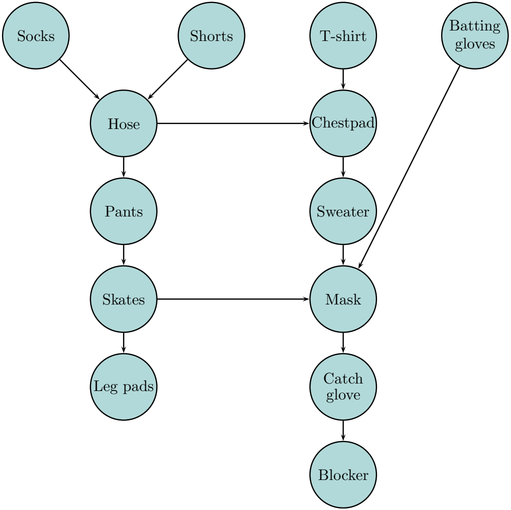
In DFS of an undirected graph, we get only tree and back edges. No forward or cross edges.

Starting from 1, either 2 discovers 3 or vice versa, therefore one of them is the other's descendant, Hence no cross edges.

Directed Acyclic Graph (DAG)

- DAG – Directed graph with no cycles.
- Good for modeling processes and structures that have a partial order:
- $a > b$ and $b > c \Rightarrow a > c$.
- But may have a and b such that neither $a > b$ nor $b > a$.
- Can always *make* a total order (either $a > b$ or $b > a$ for all $a \neq b$) from a partial order.

Directed Acyclic Graph (DAG) – Example



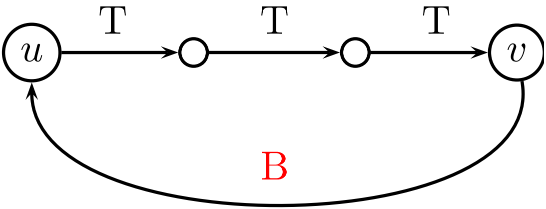
Characterizing a DAG

Lemma

A directed graph G is acyclic iff a DFS of G yields no back edges.

Proof.

$\Rightarrow$ Show that back edge $\rightarrow$ cycle:
Suppose there is a back edge (u, v). Then v is ancestor of u in depth-first forest. Therefore, there is a path $v \rightsquigarrow u$, so $v \rightsquigarrow u \rightsquigarrow v$ is a cycle. □

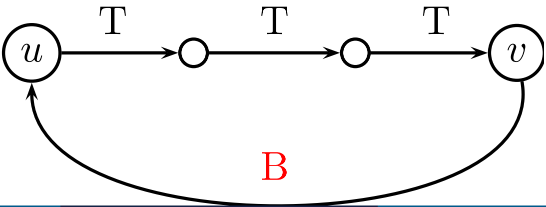


Characterizing a DAG

Proof.

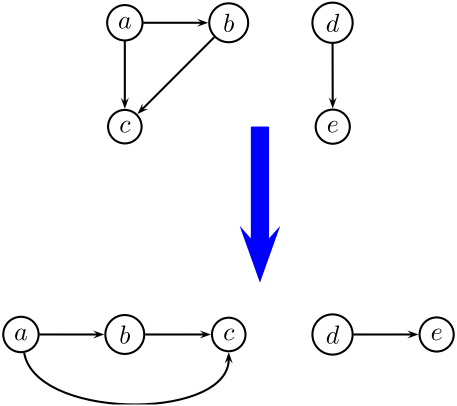
$\Rightarrow$: Show that a cycle implies a back edge.

- c : cycle in G, u : first vertex discovered in c, (v, u) : preceding edge in c.
- At time d[v], vertices of c form a white path $u \rightsquigarrow v$. Why?
- By white-path theorem, v is a descendent of u in depth-first forest.
- Therefore, (v, u) is a back edge. $\square$



Topological Sorting

- We want to "sort" a DAG.
- Think of original DAG as a partial order.
- We want a total order that extends this partial order.



Topological Sorting

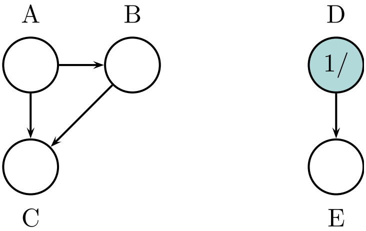
- Performed on a DAG.
- Linear ordering of the vertices of G such that if $(u, v) \in E$, then u appears somewhere before v.

TopologicalSort(G)

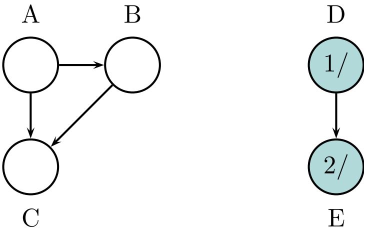
1. call DFS(G) to compute finishing times f[v] for all $v \in V$
2. as each vertex is finished, insert it onto the front of a linked list
3. return the linked list of vertices (with *decreasing* finish times)

Runtime – $\Theta(V + E)$

Topological Sorting – Example

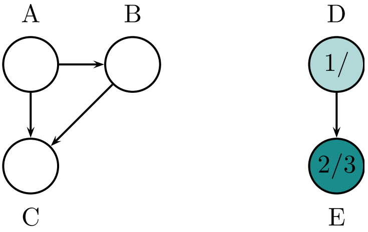


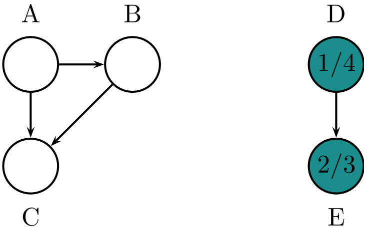
Linked list:



Linked list:

Topological Sorting – Example





Linked list:

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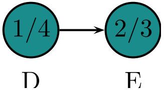
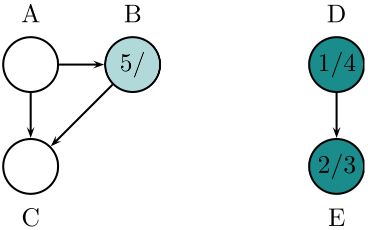
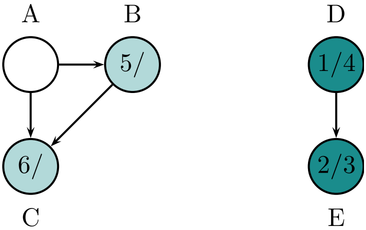
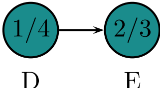


Topological Sorting – Example





Linked list:



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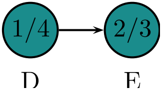
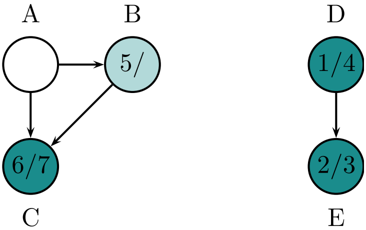
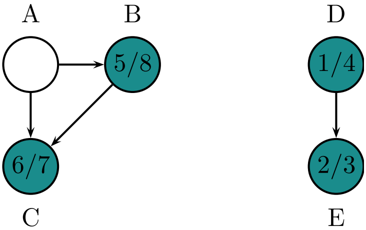


Topological Sorting – Example





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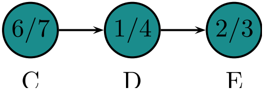

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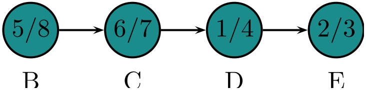
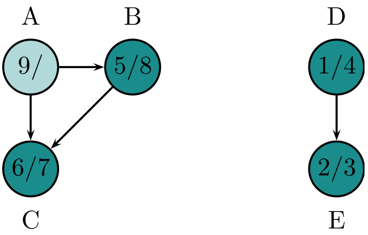
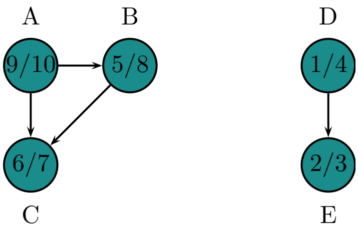


Topological Sorting – Example





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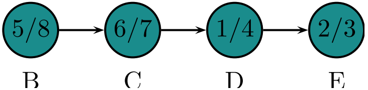

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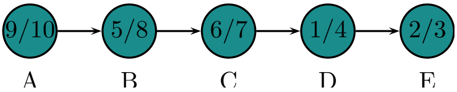
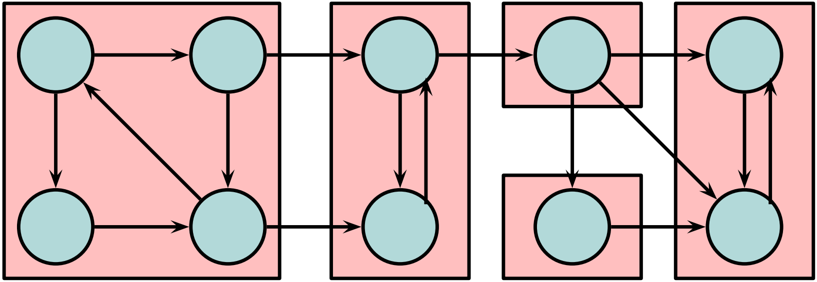


Topological Sorting – Proof of Correctness

- Just need to show if $(u, v) \in E$, then $f[v] < f[u]$.
- When we explore (u,v) then u is gray. What is the color of v?
- Is v **gray**?
- No, because then v would be ancestor of u. $\Rightarrow (u, v)$ is a back edge, which contradicts the fact that A DAG has no back edges.
- Is v **white**?
- Then becomes descendant of u.
- By parenthesis theorem, $d[u] < d[v] < f[v] < f[u]$.
- Is v **black**?
- Then v is already finished.
- Since we're exploring (u,v), we have not yet finished u.
- Therefore, $f[v] < f[u]$.

Strongly Connected Components

- G is strongly connected if every pair (u, v) of vertices in G is reachable from one another.
- A strongly connected component (SCC) of G is a maximal set of vertices $C \subseteq V$ such that for all $u, v \in C$, there is a path from u to v and from v to u.



$G^{SCC} isa DAG$

Theorem

Let C and C' be distinct SCC's in G, let u, $v \in C, u', v' \in C'$, and suppose there is a path $u \rightsquigarrow u'$ in G. Then there cannot also be a path $v' \rightsquigarrow v$ in G.

Proof.

- Suppose there is a path from v' to v in G.
- Then there are paths from u to u' to v' and from v' to v to u in G.
- Therefore, u and v' are reachable from each other, so they are not in separate SCC's.

$\square$

Transpose of a Directed Graph

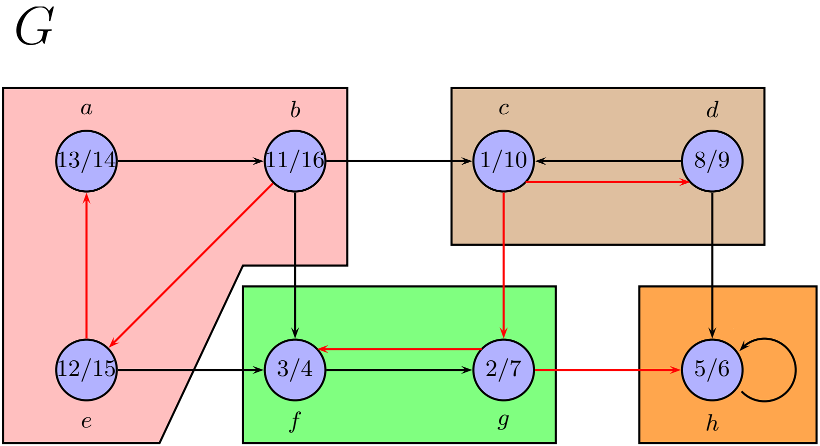
- G^T = transpose of directed G.
- $G^T = (V, E^T), E^T = (u, v) : (v, u) \in E$.
- G^T is G with all edges reversed.
- Can create G^T in $\Theta(V + E)$ time if using adjacency lists.
- G and G^T have the same SCC's. (u and v are reachable from each other in G if and only if reachable from each other in G^T).

Algorithm to Determine SCC

1. Call $DFS(G)$ to compute finishing times f [u] for all u
2. Compute G^T
3. Call $DFS(G^T)$, but in the main loop, consider vertices in order of decreasing f [u] (as computed in first DFS)
4. Output the vertices in each tree of the depth-first forest formed in second DFS as a separate SCC

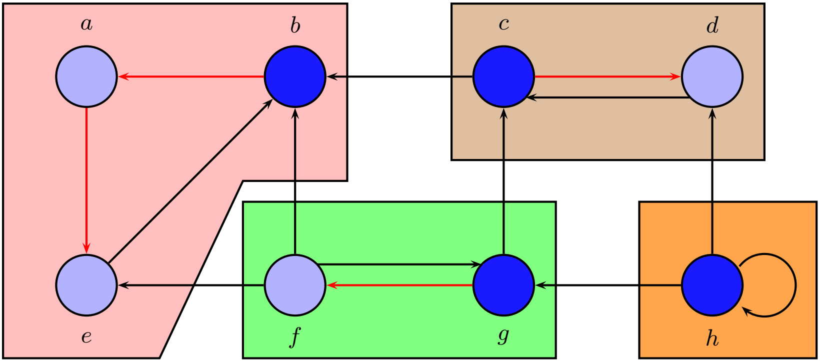
Runtime – $\Theta(V + E)$

Example

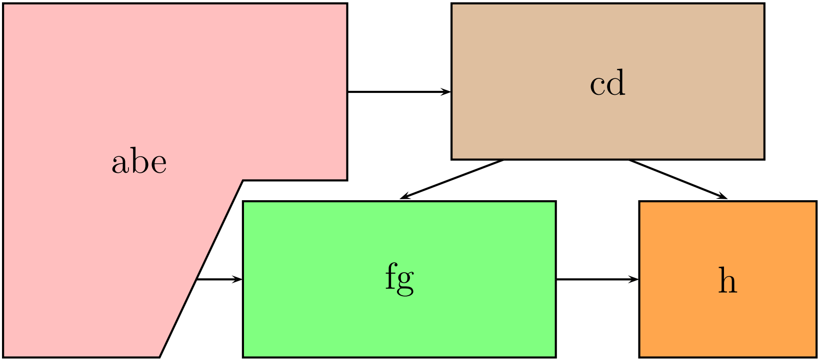


Example

G^T



Example



How Does it Work?

- Idea:
  - By considering vertices in second DFS in decreasing order of finishing times from first DFS, we are visiting vertices of the component graph in topologically sorted order.
  - Because we are running DFS on GT, we will not be visiting any v from a u, where v and u are in different components.
- Notation:
  - d[u] and f[u] always refer to first DFS.
  - Extend notation for d and f to sets of vertices $U \subseteq V$:
  - $d(U) = \min_{u \in U}\{d[u]\}$ (earliest discovery time)
  - $f(U) = \max_{u \in U}\{f[u]\}$ (latest finishing time)

SCCs and DFS finishing times

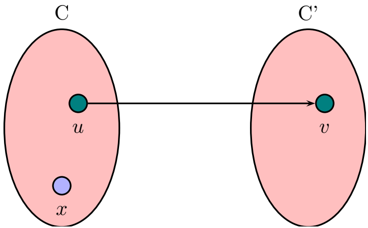
Lemma

Let C and C' be distinct SCC's in G = (V, E). Suppose there is an edge $(u, v) \in E$ *such that* $u \in C$ *and* $v \in C'$. *Then* $f(C) > f(C')$.

Case 1: $d(C) < d(C')$.

- Let x be the first vertex discovered in C.
- At time d[x], all vertices in C and C' are unvisited. Thus, there exist paths of unvisited vertices from x to all vertices in C and C'.
- All vertices in C and C' are descendants of x in depth-first tree.
- Therefore, $f[x] = f(C) > f(C')$.

□



SCCs and DFS finishing times

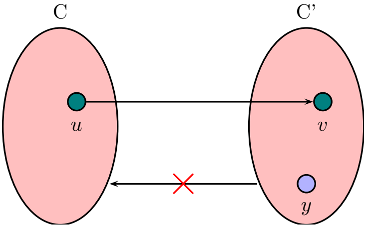
Lemma

Let C and C' be distinct SCC's in G = (V, E). Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

Case 2: $d(C) > d(C')$.

- Let y be the first vertex discovered in C'.
- At time d[y], all vertices in C' are unvisited and there is an unvisited path from y to each vertex in C" all vertices in C' become descendants of y. Again, f [y] = f (C').
- At time d[y], all vertices in C are also unvisited.

$\square$



SCCs and DFS finishing times

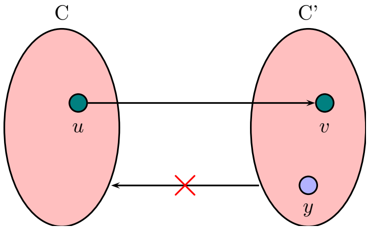
Lemma

Let C and C' be distinct SCC's in G = (V, E). Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

Case 2: $d(C) > d(C')$.

- By earlier lemma, since there is an edge (u, v), we cannot have a path from C' to C.
- So no vertex in C is reachable from y.
- Therefore, at time f[y], all vertices in C are still white.
- Therefore, for all $w \in C$, $f[w] > f[y]$, which implies that $f(C) > f(C')$.

□



SCCs and DFS finishing times

Corollary

Let C and C' be distinct SCC's in G = (V, E). Suppose there is an edge $(u, v) \in E^T$, where $u \in C$ and $v \in C'$. Then $f(C) < f(C')$.

Proof.

$(u, v) \in E^T \Rightarrow (v, u) \in E$. Since SCC's of G and G^T are the same, $f(C') > f(C)$, by former Lemma. $\square$

Correctness of SCC

- When we do the second DFS, on G^T, start with SCC C such that f(C) is maximum.
- The second DFS starts from some $x \in C$, and it visits all vertices in C.
- Corollary above says that since $f(C) > f(C')$ for all $C \neq C'$, there are no edges from C to C' in G^T.
- Therefore, DFS will visit only vertices in C.
- Which means that the depth-first tree rooted at x contains exactly the vertices of C.

Correctness of SCC

- The next root chosen in the second DFS is in SCC C' such that f(C') is maximum over all SCC's other than C.
- DFS visits all vertices in C', but the only edges out of C' go to C, which we've already visited.
- Therefore, the only tree edges will be to vertices in C'.
- We can continue the process.
- Each time we choose a root for the second DFS, it can reach only vertices in its SCC–get tree edges to these,
- Vertices in SCC's already visited in second DFS–get no tree edges to these.