CS624: Analysis of Algorithms

Final Exam – Practice

## Instructor – Nurit Haspel

1. **Greedy algorithms:**

   You drive from Boston to Washington DC. Your gas tank, when full, holds enough gas to travel m miles and you have a map that gives you the distances between gas stations on the way. Let $d_1 < d_2 < ... < d_n$ be the locations of the gas stations, where d_i is the distance from Boston to the gas station (the list is sorted). Assume that the distance between two consecutive gas stations is at most m miles.

   Your goal is to make as few stops for gas as possible along the way without getting stuck. A greedy algorithm to solve the problem is to always drive as far as possible before stopping for gas. More formally, let c_i be the destination with distance d_i from Boston:

   $S = \emptyset$

   $last = 0$

   for $(i = 1..n)$

       if $(d_i - last > m)$ // d_i is the distance of first station that's too far

           $S = S \cup c_{i-1}$ // Add the one before (the last that's not too far)

           $last = d_{i-1}$

   return S

   (a) Show the problem has the optimal substructure property. Start by: Let's say S is an **optimal** solution. Given that we stop for gas at stop at d_i, then what can you say about the the parts of the road uncovered by d_i?

   (b) Show that the greedy choice is safe. Specifically, let g be the first stop selected by the greedy algorithm (the farthest from Boston you can stop without getting stuck). Show that there is an optimal solution S that contains g as its first stop. Explain your answer.

2. (20%) **Graphs:** In this question we refer to a node as "undiscovered" if it has not been seen yet by a graph walk (colored white in class). A node is "discovered" if it has been seen but not yet done (colored gray in class). A node is "processed" if it is done (colored black in class).

   (a) (10%) Show an example of a graph $G = (V, E)$ such that a BFS walk yields $O(|V|)$ processed (gray) nodes at some given moment. Explain. **To get the full mark your example should contain at least four nodes.**

   (b) (10%) Show an example of a graph $G = (V, E)$ such that a DFS walk yields $O(|V|)$ processed (gray) nodes at some given moment. Explain. **To get the full mark your example should contain at least four nodes.**

3. (20%) **Flow:** Given the following flow graph:
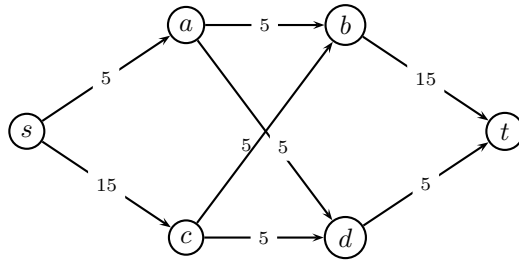
Show the maximum flow for this graph. No need to show the residual graphs. Write the augmenting paths you find, the value of the flow for each path and the value of the overall maximum flow.

4. (25%) **NP:** The directed Hamiltonian Path (HAM-Path) is the problem of finding a simple path that goes through every vertex in a directed graph once. To show that the problem is NP-complete we use a reduction from the directed Hamiltonian cycle problem (HAM-Cycle) – we start from an instance $G = (V, E)$ to the HAM-cycle problem. We create a directed graph $H = (V', E')$ as follows: Select an arbitrary node $u \in V$ and split it to two nodes, u_{in} and u_{out}. Every edge (u, v) in G will now become (u_{out}, v) in H and every edge (v, u) becomes (v, u_{in}) in H. The other vertices and edges remain unchanged.

(a) (6%) Show that directed HAM-Path is in NP.

(b) (7%) Show that the reduction described above is polynomial.

(c) (7%) Show that the graph G has a HAM-Cycle iff H has a HAM-Path. Don't forget the two directions.