THEORY OF COMPUTATION
Problem session - 1

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UMB

1. Let $\mathcal{P}$ be the program

$$\begin{array}{ll} & \text{IF } X \neq 0 \text{ GOTO } A \\ [A] & X \leftarrow X + 1 \\ & \text{IF } X \neq 0 \text{ GOTO } A \\ [A] & Y \leftarrow Y + 1 \end{array}$$

   What is $\Psi_{\mathcal{P}}^{(1)}(x)$?
2. Show that for every partially computable function $f(x_1, \ldots, x_n)$ there is a number $m \geq 0$ such that f is computed by infinitely many programs of length m.
3. Show by constructing a program that the predicate $x_1 \leqslant x_2$ is computable.
4. Let $P(x)$ be a computable predicate. Show that the function $E_P(r)$ defined by

$$E_P(r) = \begin{cases} 1 & \text{if there are at least } r \text{ numbers } n \text{ such that } P(n) = 1 \\ \uparrow & \text{otherwise} \end{cases}$$

   is partially computable.

1. Let $\mathcal{P}$ be the program

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What is $\Psi_{\mathcal{P}}^{(1)}(x)$?
Solution: Note that regardless of the initial value of X, the statement $X \leftarrow X + 1$ is repeated indefinitely. Thus, $\Psi_{\mathcal{P}}^{(1)}(x_1) \uparrow$ for every $x \in \mathbb{N}$, hence $\Psi_{\mathcal{P}}^{(1)}(x_1) = \emptyset$.

2. Show that for every partially computable function $f(x_1, \ldots, x_n)$ there is a number $m \geq 0$ such that f is computed by infinitely many programs of length m.
Solution: If $\mathcal{P}$ is a program that computes f, let Z_k be the first intermediate variable that does not occur in $\mathcal{P}$. Suppose that the length of $\mathcal{P}$ is ℓ.
Consider the program $\mathcal{R}_h$ that has the length $m = \ell + 1$:

$$\begin{array}{l} Z_h \leftarrow Z_h \\ \mathcal{P} \end{array}$$

The family of programs $\{\mathcal{R}_h \mid h \geqslant k\}$ is infinite and all these programs compute f.

3. Show by constructing a program that the predicate $x_1 \leqslant x_2$ is computable.

Solution:

$$
\begin{array}{ll}
[L] & X_1 \leftarrow X_1 - 1 \\
 & X_2 \leftarrow X_2 - 1 \\
 & \text{IF } X_1 = 0 \text{ GOTO } L_1 \\
 & \text{IF } X_2 = 0 \text{ GOTO } L_2 \\
 & \text{GOTO } L \\
[L_1] & Y \leftarrow 1 \\
 & \text{GOTO } E \\
[L_2] & Y \leftarrow 0
\end{array}
$$

4. Let $P(x)$ be a computable predicate. Show that the function $E_P(r)$ defined by

$$E_P(r) = \begin{cases} 1 & \text{if there are at least } r \text{ numbers } n \text{ such that } P(n) = 1 \\ \uparrow & \text{otherwise} \end{cases}$$

is partially computable.
Solution:

$$\begin{array}{ll} & Z_1 \leftarrow 0 \\ [L] & \text{IF } Z_1 \geqslant r \text{ GOTO } L_2 \\ & \text{IF } \sim P(Z_1) \text{ GOTO } L_1 \\ & Z_1 \leftarrow Z_1 + 1 \\ [L_1] & \text{GOTO } L \\ [L_2] & Y \leftarrow 1 \end{array}$$