THEORY OF COMPUTATION
Problem session - 3

Prof. Dan A. Simovici

UMB

1. For any unary function $f(x)$, the n^{th} iteration of f, written f^n, is

$$f^n(x) = f(\cdots f(x) \cdots),$$

where f is composed with itself n times on the right side of the equality. Note that $f^0(x) = x$. Let $h_f(n, x) = f^n(x)$. Show that if f is primitive recursive, then h_f is also primitive recursive.
2. Let $f(0) = 0$, $f(1) = 1$, $f(2) = 2^2$, $f(3) = 3^{3^3}$, etc. In general, $f(n)$ is written as a stack n high, of ns as exponents. Show that f is primitive recursive.
3. Let g be a primitive recursive function and let $f(0, x) = g(x)$, $f(n + 1, x) = f(n, f(n, x))$. Prove that $f(n, x)$ is primitive recursive.

Problem 1: For any unary function $f(x)$, the n^{th} iteration of f, written f^n, is

$$f^n(x) = f(\cdots f(x) \cdots),$$

where f is composed with itself n times on the right side of the equality. Note that $f^0(x) = x$. Let $h_f(n, x) = f^n(x)$. Show that if f is primitive recursive, then h_f is also primitive recursive.

Solution for Problem 1: Note that

$$\begin{aligned} h_f(0,x) &= x, \\ h_f(n+1,x) &= f^{n+1}(x) = f(f^n(x)) = f(h_f(n,x)). \end{aligned}$$

The second equality can be written as

$$h_f(n+1,x) = F(n, h_f(n,x), x),$$

where $F(n,z,x) = f(z) = f(p_2^3(n,z,x))$. This is a definition by primitive recursion of h_f. Thus, if f is primitive recursive, then so is h_f.

Problem 2: Let $f(0) = 0$, $f(1) = 1$, $f(2) = 2^2$, $f(3) = 3^{3^3}$, etc. In general, $f(n)$ is written as a stack n high, of ns as exponents. Show that f is primitive recursive.

Solution for Problem 2: Define the function g as

$$g(n,k) = n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}$$

containing k exponents. The function g is primitive recursive because

$$\begin{aligned} g(n,0) &= 1, \\ g(n,k+1) &= n^{g(n,k)}. \end{aligned}$$

This implies the primitive recursiveness of f because $f(n) = g(n,n)$.

Problem 3: Let g be a primitive recursive function and let $f(0,x) = g(x)$, $f(n+1,x) = f(n, f(n,x))$. Prove that $f(n,x)$ is primitive recursive.

Solution: Note that we can write

$$\begin{aligned}
f(0,x) &= g(x);\\
f(1,x) &= f(0, f(0,x)) = f(0, g(x)) = g(g(x)) = g^2(x);\\
f(2,x) &= f(1, f(1,x)) = f(1, g^2(x)) = f(0, f(0, g^2(x)) = f(0, g^3(x))\\
&= g^4(x).
\end{aligned}$$

In general, we have $f(n,x) = g^{2^n}(x)$ (by induction on n). The basis step, $n = 0$ is immediate. If this this holds for n,

$$\begin{aligned}
f(n+1,x) &= f(n, f(n,x)) = f(n, g^{2^n}(x))\\
&\quad \text{(by the inductive hypothesis)}\\
&= g^{2^n}(g^{2^n}(x))\\
&\quad \text{(by another application of the inductive hypothesis)}\\
&= g^{2^n+2^n}(x) = g^{2^{n+1}}(x),
\end{aligned}$$

which proves that $f(n,x) = g^{2^n}(x)$.

Note that the function $h(\ell, x) = g^{\ell}(x)$ is primitive recursive because

$$\begin{aligned} h(0, x) &= x, \\ h(\ell + 1, x) &= g(h(\ell, x)). \end{aligned}$$

Since $f(n, x) = h(2^n, x)$ it follows that f is primitive recursive.