THEORY OF COMPUTATION
Problem session - 4

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UMB

1. Let COMP be the class of functions obtained from the initial functions be a finite sequence of compositions.
    1. Show that for every function $f(x_1, \dots, x_n)$ in COMP, either $f(x_1, \dots, x_n) = k$ for some constant k, or $f(x_1, \dots, x_n) = x_i + k$ for some $1 \leqslant i \leqslant k$.
    2. An n-ary function f is monotone if for all n-tuples $(x_1, \dots, x_n), (y_1, \dots, y_n)$ such that $x_i \leqslant y_i$ for $1 \leqslant i \leqslant n$ we have $f(x_1, \dots, x_n) \leqslant f(y_1, \dots, y_n)$. Prove that every function in COMP is monotone.
    3. Show that COMP is a proper subset of the class of primitive recursive functions.
2. Let us call a predicate trivial if it always TRUE or always FALSE. Show that no nontrivial predicate belongs to COMP.
3. Let $\pi(x)$ be the number of primes that are less or equal to x. Show that $\pi(x)$ is primitive recursive.

Problem 1: Let COMP be the class of functions obtained from the initial functions be a finite sequence of compositions.

1. Show that for every function $f(x_1, \dots, x_n)$ in COMP, either $f(x_1, \dots, x_n) = k$ for some constant k, or $f(x_1, \dots, x_n) = x_i + k$ for some i, where $1 \leqslant i \leqslant k$.
2. An n-ary function f is monotone if for all n-tuples $(x_1, \dots, x_n), (y_1, \dots, y_n)$ such that $x_i \leqslant y_i$ for $1 \leqslant i \leqslant n$ we have $f(x_1, \dots, x_n) \leqslant f(y_1, \dots, y_n)$. Prove that every function in COMP is monotone.
3. Show that COMP is a proper subset of the class of primitive recursive functions.

Solution for Problem 1: The conditions of Part 1 are clearly satisfied by the initial functions s, n, and u_i^n. Suppose that $f : \mathbb{N}^n \longrightarrow \mathbb{N}$ and $g_1, \dots, g_n : \mathbb{N}^p \longrightarrow \mathbb{N}$ are functions in COMP such that $f(x_1, \dots, x_n) = x_i + k$ and $g_\ell(x_1, \dots, x_p) = x_{m_\ell} + k_\ell$. Then, we have either

$$f(g_1(x_1, \dots, x_p), \dots, g_n(x_1, \dots, x_p)) = k,$$

if f itself is a constant, or

$$f(g_1(x_1, \dots, x_p), \dots, g_n(x_1, \dots, x_p)) = g_i(g_1(x_1, \dots, x_p) + k.$$

Continuation of Solution for Problem 1: Since $g_i \in$ COMP we have either $g_i(x_1, \ldots, x_p) = k_i$ for some constant k_i, or $g_i(x_1, \ldots, x_n) = x_{h_i} + k_i$. In the first case,

$$f(g_1(x_1, \ldots, x_p), \ldots, g_n(x_1, \ldots, x_p)) = k,$$

while in the second,

$$f(g_1(x_1, \ldots, x_p), \ldots, g_n(x_1, \ldots, x_p)) = x_{h_i} + k_i + k.$$

Thus, every function in COMP satisfies the conditions of Part 1.

Hint for Part 2: Observe that the initial functions are monotone. Then, suppose that each of the functions $f : \mathbb{N}^n \longrightarrow \mathbb{N}$ and $g_1, \ldots, g_n : \mathbb{N}^p \longrightarrow \mathbb{N}$ are monotone, and verify that $f(g_1, \ldots, g_n)$ is monotone.

Hint for Part 2: You need to show only that there is a primitive recursive function that is not monotone. Can you think of an example of such a function?

Problem 2: Let us call a predicate trivial if it always TRUE or always FALSE. Show that no nontrivial predicate belongs to COMP.

Solution for Problem 2: A predicate in COMP must satisfy the conditions established in Part 1 of Problem 1 for functions, namely, $P(x_1, \ldots, x_n) = k$ or $P(x_1, \ldots, x_n) = x_i + k$ for some i. The second condition can be excluded because the set of values of a predicate is $\{0, 1\}$. Therefore, if P is in COMP it can be only a constant (0 or 1).

Problem 3: Let $\pi(x)$ be the number of primes that are less or equal to x. Show that $\pi(x)$ is primitive recursive.
Solution for Problem 3: Note that $\pi(x) = \sum_{p=0}^{x} \text{Prime}(p)$.