# THEORY OF COMPUTATION
## Problem session - 5

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UMB

1. Problems
2. Solutions

1. Show that the function $f(x) = \lfloor \log_2(x+1) \rfloor$ is primitive recursive.
2. Let $h(x)$ be the integer $n$ such that $n \leqslant \sqrt{2}x \leqslant n+1$. Prove that $h$ is primitive recursive.
3. Let $\mathrm{lcm}(x, y)$ be the least common multiple of $x$ and $y$. Prove that $\mathrm{lcm}$ is primitive recursive.
4. Let $\gcd(x, y)$ be the greatest common divisor of $x$ and $y$. Prove that $\gcd$ is primitive recursive.

An useful special case of bounded minimalization: when defining functions by bounded minimalization one could take $n = 0$. So, if $P$ is a predicate that depends on one variable, then the function defined by bounded minimalization from $P$ is

$$f(y) = \begin{cases} \min_{t \leqslant y} P(t) & \text{if there exists } t \leqslant y \text{such that } P(t) = TRUE, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 1: Let $f(x) = \lfloor \log_2(x+1) \rfloor$. Prove that $f$ is primitive recursive.

Solution of Problem 1: Let $t = \lfloor \log_2(x+1) \rfloor$. We have $t = \lfloor \log_2(x+1) \rfloor \leqslant \log_2(x+1) < t+1$. Therefore, $t+1$ is the smallest number that is larger than $\log_2(x+1)$. This is equivalent to saying that $t+1$ is the smallest number such that $\lfloor \log_2(x+1) \rfloor < t+1$. Therefore, $t+1$ is the smallest number such that $x+1 < 2^{t+1}$. Since the logarithm is a stricly increasing function, we have

$$t \leqslant \log_2(x+1) < t+1 < x+1.$$

This implies

$$f(x) = \lfloor \log_2(x+1) \rfloor = \min_{t \leqslant x+1} [2^{t+1} \geqslant x+1],$$

so $f$ is primitive recursive.

Problem 2: Let $h(x)$ be the integer $n$ such that $n \leqslant \sqrt{2}x \leqslant n+1$. Prove that $h$ is primitive recursive.
Solution for Problem 2: Let $t = \lfloor \sqrt{2}x \rfloor$. We have

$$t = \lfloor \sqrt{2}x \rfloor \leqslant \sqrt{2}x < t+1.$$

Therefore, $t+1$ is the smallest number that is larger than $\sqrt{2}x$. Thus, $h(x) = \min_{t \leqslant x}(2x^2 < (t+1)^2)$.

Problem 3: Let $\mathrm{lcm}(x, y)$ be the least common multiple of $x$ and $y$. Prove that $\mathrm{lcm}$ is primitive recursive.
Solution for Problem 4: Observe that we can write

$$\mathrm{lcm}(x, y) = \min_{t<xy}[(x|t)\&(y|t)],$$

so $\mathrm{lcm}$ is obtained by bounded minimalization.

Problem 4: Let gcd$(x, y)$ be the greatest common divisor of $x$ and $y$. Prove that gcd is primitive recursive.
Solution for Problem 5: Observe that we can write

$$\text{gcd}(x, y) = \lfloor xy/\mathrm{lcm}(x, y) \rfloor,$$

so gcd is primitive recursive.