# THEORY OF COMPUTATION
## Problem session - 6

Prof. Dan A. Simovici

UMB

Problem 1: Prove that the functions min and max where

$$\min(x, y) = \begin{cases} x & \text{if } x \leqslant y, \\ y & \text{if } x \geqslant y, \end{cases} \quad \text{and} \quad \max(x, y) = \begin{cases} y & \text{if } x \leqslant y, \\ x & \text{if } x \geqslant y, \end{cases}$$

are primitive recursive.

Problem 2: Let

$$\begin{aligned} h_1(x,0) &= f_1(x), \\ h_2(x,0) &= f_2(x), \\ h_1(x,t+1) &= g_1(x,h_1(x,t),h_2(x,t)), \\ h_2(x,t+1) &= g_2(x,h_1(x,t),h_2(x,t)). \end{aligned}$$

Prove that if $f_1, f_2, g_1, g_2$ all belong to some PRC class $\mathcal{C}$, then $h_1, h_2$ do also.

Problem 3: Let $\text{trim} : \mathbb{N} \longrightarrow \mathbb{N}$ be the function defined as follows: if $z = [x_1, \ldots, x_{n-1}, x_n]$, then $\text{trim}(z) = [x_1, \ldots, x_{n-1}]$. For the special case when $z = 0$, we define $\text{trim}(z) = 0$. Prove that trim is primitive recursive.

Problem 4: The function insert : $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ takes the code of an sorted list of numbers $x$, and number $y$, inserts $y$ in to the list in the correct position and returns the code of the new list. Prove that insert is primitive recursive.

Problem 1: Prove that the functions min and max where

$$\max(x,y) = \begin{cases} y & \text{if } x \leqslant y, \\ x & \text{if } x \geqslant y, \end{cases} \text{ and } \min(x,y) = \begin{cases} x & \text{if } x \leqslant y, \\ y & \text{if } x \geqslant y, \end{cases}$$

are primitive recursive.

**Solution:** Note that

$$\max(x,y) = \lceil (x + y + |x - y|)/2 \rceil$$

and

$$\min(x,y) = \lceil (x + y \dot{-} |x - y|)/2 \rceil,$$

which implies the primitive recursiveness.

Problem 2: Let

$$\begin{aligned} h_1(x,0) &= f_1(x), \\ h_2(x,0) &= f_2(x), \\ h_1(x,t+1) &= g_1(x, h_1(x,t), h_2(x,t)), \\ h_2(x,t+1) &= g_2(x, h_1(x,t), h_2(x,t)). \end{aligned}$$

Prove that if $f_1, f_2, g_1, g_2$ all belong to some PRC class $\mathcal{C}$, then $h_1, h_2$ do also.

**Solution:** Define the function $F(x,t) = \langle h_1(x,t), h_2(x,t)\rangle$.
Clearly,

$$F(x,0) = \langle f_1(x), f_2(x)\rangle.$$

Also,

$$\begin{aligned} F(x,t+1) &= \langle h_1(x,t+1), h_2(x,t+1)\rangle \\ &= \langle g_1(x, h_1(x,t), h_2(x,t)), g_2(x, h_1(x,t), h_2(x,t))\rangle \\ &= \langle g_1(x, \ell(F(x,t)), g_2(x, r(F(x,t))\rangle, \end{aligned}$$

which is a definition be primitive recursion of $F$. Thus, $F \in \mathcal{C}$. Since $h_1(x,t) = \ell(F(x,t))$ and $h_2 = r(F(x,t))$ and $\ell, r, F \in \mathcal{C}$, it follows that $h_1, h_2 \in \mathcal{C}$.

Problem 3: Let trim : $\mathbb{N} \longrightarrow \mathbb{N}$ be the function defined as follow: if $z = [x_1, \ldots, x_{n-1}, x_n]$, then $\mathrm{trim}(z) = [x_1, \ldots, x_{n-1}]$. For the special case when $z = 0$, we define $\mathrm{trim}(0) = 0$. Prove that trim is primitive recursive.

**Solution:** The primitive recursiveness of trim results from the equality

$$\mathrm{trim}(z) = \min_{u \leqslant z} \bigwedge_{i=1}^{\mathrm{Lt}(z)-1} [(u)_i = (z)_i].$$

Problem 4: The function insert : $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$ takes the code of an sorted list of numbers $x$ and number $y$, inserts $y$ in to the list in the correct position and returns the code of the new list. Prove that insert is primitive recursive.

**Solution:** Let $\mu(x, y)$ be the largest position in $x$ whose component is smaller than $y$. We have $\mu(x, y) = \max_{j \leqslant \mathrm{Lt}(x)} (x)_j < y$. Clearly, $\mu$ is primitive recursive. Then,

$$\mathrm{insert}(x, y) = \prod_{i=1}^{\mu(x,y)} p_i^{(x)_i} \cdot p_{\mu(x,y)+1}^{y} \prod_{k=\mu(x,y)+2} p_k^{(x)_{k-1}},$$

which proves that insert is primitive recursive.

For example, suppose that $x = [1, 4, 7, 8, 10]$ and we need to compute insert$(x, 6)$. We have $\mu(x, 6) = 2$ because $(x)_1 = 1 < 6$, $(x)_2 = 4 < 6$ but $(x)_3 = 7 \not< 6$. Therefore,

$$\mathrm{insert}(x, 6) = p_1^1 p_2^4 p_3^6 p_4^7 p_5^8 p_6^{10}.$$