THEORY OF COMPUTATION
Recursively Enumerable Sets - 11 part 2

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UMB

Recall that by the Enumeration Theorem, the collection of r.e. sets can be written as

$$W_1, W_2, \ldots, W_n, \ldots$$

where $W_n = \{x \in \mathbb{N} \mid \Phi(x, n)\downarrow\}$.

Let K be the set of all numbers n such that the program number n eventually halts on number n.

Definition

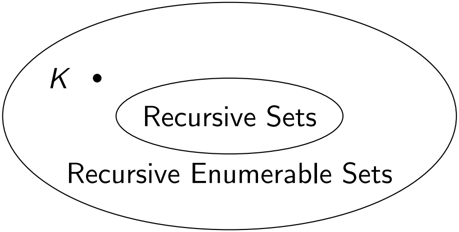
The set K is defined as:

$$K = \{n \in \mathbb{N} \mid n \in W_n\}.$$

Note that $n \in W_n$ if and only if $\Phi(n, n) \downarrow$ if and only if $\text{HALT}(n, n) = \text{TRUE}$.

Theorem

The set K is r.e. but not recursive.

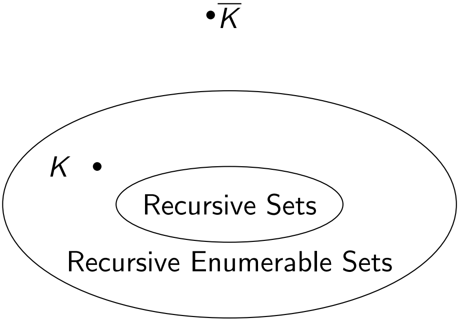


Proof.

Note that $K = \{n \in \mathbb{N} \mid \Phi(n, n) \downarrow\}$, Φ is partially computable, hence K is r.e.
If $\overline{K}$ were r.e, by the Enumeration Theorem we would have $\overline{K} = W_i$ for some i. Then,

$$i \in K \Leftrightarrow i \in W_i \Leftrightarrow i \in \overline{K},$$

which is a contradiction. Therefore, $\overline{K}$ is not r.e., which implies that K is not recursive. □


$\bullet\overline{K}$
K $\bullet$
Recursive Sets
Recursive Enumerable Sets

An alternative characterization of r.e. sets is provided next.

Theorem

Projection Theorem *Let B be an r.e. set. There exists a primitive recursive predicate $R(x, t)$ such that:*

$$B = \{x \in \mathbb{N} \mid (\exists t)R(x, t)\}.$$

Note that this is unbounded existential quantification!

Proof.

Let $B = W_n$. Then,

$$B = \{x \in \mathbb{N} \mid (\exists t)\mathrm{STP}^{(1)}(x, n, t)\}$$

and STP is primitive recursive. The role of $R(x, t)$ is played by $\mathrm{STP}^{(1)}(x, n, t)$. □

Theorem

Let S be a non-empty r.e. set. Then, there is a primitive recursive function $f(u)$ such that

$$S = \{f(n) \mid n \in \mathbb{N}\} = \{f(0), f(1), \ldots\}.$$

That is, S is the range of f.

Proof.

By a previous theorem, $S = \{x \in \mathbb{N} \mid (\exists t)R(x,t)\}$, where R is a primitive recursive predicate. Let x_0 be some fixed member of S (for example the smallest). Let f the primitive recursive function:

$$f(u) = \begin{cases} \ell(u) & \text{if } R(\ell(u), r(u)), \\ x_0 & \text{otherwise.} \end{cases}$$

Each value of $f(u)$ is in S, since $x_0 \in S$, while if $R(\ell(u), r(u))$ is TRUE, then $(\exists t)R(\ell(u), t)$ is TRUE, which implies that $f(u) = \ell(u) \in S$.
Conversely, if $x \in S$, then $R(x, t_0)$ is TRUE for some t_0. Then, $f(\langle x, t_0\rangle) = \ell(\langle x, t_0\rangle) = x$, so that $x = f(u)$ for $u = \langle x, t_0\rangle$. □

Theorem

Let $f(x)$ be a partially computable function and let $S = \{f(x) \mid f(x)\downarrow\}$. In other words, S is the range of f. Then, S is r.e.

Proof.

Let

$$g(x) = \begin{cases} 0 & \text{if } x \in S, \\ \uparrow & \text{otherwise.} \end{cases}$$

Since $S = \{x \mid g(x)\downarrow\}$ it suffices to show that the fact that partially computability of f implies that g is partially computable. □

Proof cont'd

Proof.

Let $\mathcal{P}$ be a program that computes f and $\#(\mathcal{P}) = p$. Then, g is computed by

$$
\begin{array}{ll}
[A] & \text{IF } \; STP^{(1)}(Z, p, T) \text{ GOTO } B \\
 & V \leftarrow f(Z) \\
 & \text{IF } V = X \text{ GOTO } E \\
[B] & Z \leftarrow Z + 1 \\
 & \text{IF } Z \leqslant T \text{ GOTO } A \\
 & T \leftarrow T + 1 \\
 & Z \leftarrow 0 \\
 & \text{GOTO } A
\end{array}
$$

Note that $V \leftarrow f(Z)$ is entered only when the step-counter test has already determined that f is defined. □

Theorem

Suppose that $S \neq \emptyset$. The following statements are equivalent:

1. *S is r.e.;*
2. *S is the range of a primitive recursive function;*
3. *S is the range of a recursive function;*
4. *S is the range of a partial recursive function;*

This follows from previous theorems: the implication (1) $\Rightarrow$ (2) follows from the theorem on Slide 10. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious. The implication (4) $\Rightarrow$ (1) follows from the theorem on Slide 12.

The parameter theorem is also known as the smn-theorem.

Theorem

For each $n, m > 0$ there is a primitive recursive function $S_m^n(u_1, \ldots, u_n, y)$ such that

$$\begin{aligned}&\Phi^{(m+n)}(x_1, \ldots, x_m, u_1, \ldots, u_n, y)\\ = \quad &\Phi^{(m)}(x_1, \ldots, x_m, S_m^n(u_1, \ldots, u_n, y)).\end{aligned}$$

Proof.

The proof is by induction on n, the number of arguments $u_1, \ldots, u_n$ packed into S.
For $n = 1$, the base case, we need to show that there is a function S_m^1 such that

$$\begin{aligned} &\Phi^{(m+n)}(x_1, \ldots, x_m, u_1, y) \\ &\quad = \Phi^{(m)}(x_1, \ldots, x_m, S_m^1(u_1, y)). \end{aligned}$$

Here $S_m^1(u_1, y)$ must be the number of a program which, given m inputs $x_1, \ldots, x_m$ computes the same value as the program y does when given the $m + 1$ inputs $x_1, \ldots, x_m, u_1$.
Let $\mathcal{P}$ be the program with $\#(\mathcal{P}) = y$. Then, $S_m^1(u_1, y)$ can be taken as the number of a program which first gives X_{m+1} the value u_1 and then proceeds to execute $\mathcal{P}$. □

Proof cont'd

Proof.

So the new program begins with

$$\left.\begin{array}{c} X_{m+1} \leftarrow X_{m+1} + 1 \\ \vdots \\ X_{m+1} \leftarrow X_{m+1} + 1 \end{array}\right\} u_1$$

Note that the code of $X_{m+1} \leftarrow X_{m+1} + 1$ is

$$\langle 0, \langle 1, 2m+1 \rangle \rangle = 16m + 10.$$

□

Proof cont'd

Proof.

So, we may take

$$S_m^1(u_1, y) = \left(\prod_{i=1}^{u_1} p_i\right)^{16m+10} \prod_{j=1}^{\mathrm{Lt}(y+1)} p_{u_1+j}^{(y+1)_j} \dot{-} 1,$$

which is a primitive recursive function.
Note that the numbers of the instructions of $\mathcal{P}$ which appear as exponents in the prime power factorization of $y+1$ have been shifted to the primes $p_{u_1+1}, p_{u_1+2}, \ldots, p_{u_1+\mathrm{Lt}(y+1)}$. □

Proof cont'd

Proof.

Suppose now that the result holds for $n = k$. Then, we have

$$\begin{aligned}
&\Phi^{(m+k+1)}(x_1, \ldots, x_m, u_1, \ldots, u_k, u_{k+1}, y) \\
&\quad = \quad \Phi^{(m+k)}(x_1, \ldots, x_m, u_1, \ldots, u_k, S^1_{m+k}(u_{k+1}, y)) \\
&\quad = \quad \Phi^{m}(x_1, \ldots, x_m, S^k_m(u_1, \ldots, u_k, S^1_{m+k}(u_{k+1}, y))),
\end{aligned}$$

using the result for $n = 1$ and the induction hypothesis. Now, we can define

$$S^{k+1}_m(u_1, \ldots, u_k, u_{k+1}) = S^k_m(u_1, \ldots, u_k, S^1_{m+k}(u_{k+1}, y)).$$

□

Example

Using the smn theorem we can show the existence of a primitive recursive function g such that $\Phi_u(\Phi_v(x)) = \Phi_{g(u,v)}(x)$. This means that

$$\Phi_u(\Phi_v(x)) = \Phi(\Phi(x, v), u),$$

so $\Phi_u(\Phi_v(x)) = \Phi(x, u, v, z_0)$ is a partially computable function of x, u, v. Hence

$$\Phi_u(\Phi_v(x)) = \Phi(x, S_1^3(u, v, z_0)),$$

and $g(u, v) = S_1^3(u, v, z_0)$.