

THEORY OF COMPUTATION

Pairing Functions and Gödel Numbers - 7

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1 Pairing Functions

2 Encoding and Decoding Finite Sequence of Numbers

The purpose of this section is the study of an encoding device that makes use of primitive recursive functions.

Define the primitive recursive function

$$\langle x, y \rangle = 2^x(2y + 1) \div 1.$$

Note that $2^x(2y + 1) \neq 0$, so

$$\langle x, y \rangle + 1 = 2^x(2y + 1).$$

Theorem

If z is any given number in \mathbb{N} , there is a unique solution x, y to the equation $\langle x, y \rangle = z$.

Proof.

Note that x is the largest number such that $2^x | (z + 1)$ and y is the solution of the equation $2y + 1 = \frac{z+1}{2^x}$. Note that the equation has a solution in y because $\frac{z+1}{2^x}$ must be odd. \square

Example

The equation $\langle x, y \rangle = 39$ amounts to $2^x(2y + 1) \div 1 = 39$, or $2^x(2y + 1) = 40$. Since $40 = 2^3 \cdot 5$ we have $x = 3$ and $2y + 1 = 5$, so $y = 2$.

The equation $\langle x, y \rangle = z$ defines the functions $x = \ell(z)$ and $y = r(z)$. Since $x, y < z + 1$, we have

$$\ell(z) \leq z \text{ and } r(z) \leq z.$$

Theorem

The functions $\langle x, y \rangle$, $\ell(z)$, and $r(z)$ have the following properties:

- $\ell(\langle x, y \rangle) = x$, $r(\langle x, y \rangle) = y$;
- $\langle \ell(z), r(z) \rangle = z$;
- $\ell(z), r(z) \leq z$;
- $\langle x, y \rangle$, $\ell(z)$, and $r(z)$ are primitive recursive.

Proof.

The equalities

$$\ell(z) = \min_{x \leq z} [(\exists y)_{\leq z} (z = \langle x, y \rangle)],$$

$$r(z) = \min_{y \leq z} [(\exists x)_{\leq z} (z = \langle x, y \rangle)],$$

show that both $\ell(z)$ and $r(z)$ are primitive recursive. □

Note that $\langle x, y \rangle = z$ if and only if $x = \ell(z)$ and $y = r(z)$.

The goal of this section is to obtain primitive recursive functions that encode and decode arbitrary finite sequences of numbers. The idea was introduced by Gödel.

Definition

The *Gödel number of the sequence* (a_1, \dots, a_n) is the number

$$[a_1, \dots, a_n] = \prod_{i=1}^n p_i^{a_i}.$$

Example

The Gödel number of the sequence $(3, 1, 5, 4, 6)$ is the number

$$[3, 1, 5, 4, 6] = 2^3 \cdot 3^1 \cdot 5^5 \cdot 7^4 \cdot 11^6.$$

For each fixed n the function $[a_1, \dots, a_n]$ is primitive recursive.

Kurt Friedrich Gödel was born on April 28, 1906 and died on January 14, 1978. **Gödel** was a logician, mathematician, and philosopher and is considered along with Aristotle and Gottlob Frege to be one of the most significant logicians in history. He had an immense effect upon scientific and philosophical thinking in the 20th century, a time when others such as Bertrand Russell, Alfred North Whitehead, and David Hilbert were analyzing the use of logic and set theory to understand the foundations of mathematics pioneered by Georg Cantor.

Theorem

If $[a_1, \dots, a_n] = [b_1, \dots, b_n]$, then $a_i = b_i$ for $1 \leq i \leq n$.

Proof.

This statement is a direct consequence of the unique factorization of a number into primes. □

Note that

$$[a_1, \dots, a_n] = [a_1, \dots, a_n, 0]$$

because $p_{n+1}^0 = 1$. Similarly, we have

$$[a_1, \dots, a_n] = [a_1, \dots, a_n, 0, 0, \dots, 0].$$

However, if one adjoins 0 to the **left** of sequence the result changes. For example, we have

$$[2, 3] = 2^2 \cdot 3^3 = 108,$$

but

$$[0, 2, 3] = 2^0 \cdot 3^2 \cdot 5^3 = 1125.$$

Define the primitive recursive function $(x)_i$ so that if

$$x = [a_1, a_2, \dots, a_n],$$

then $(x)_i = a_i$. This is a **primitive recursive function** because

$$(x)_i = \min_{t \leq x} (\sim p_i^{t+1} | x).$$

Note that $(x)_0 = 0$ and $(0)_i = 0$ for all i .

Define the primitive recursive function Lt (stands for *length*)

$$Lt(x) = \min_{i \leq x} ((x)_i \neq 0 \& (\forall j)_{\leq x} (j \leq i \vee (x)_j = 0)).$$

Example

If $x = 20$, then $x = 2^2 \cdot 3^0 \cdot 5^1 = [2, 0, 1]$, then

$$(x)_1 = 2, (x)_3 = 1, (x)_4 = (x)_5 = \dots = 0,$$

so $Lt(20) = 3$. Also $Lt(0) = Lt(1) = 0$.

If $x > 1$ and $Lt(x) = n$, then p_n divides x but no prime greater than p_n divides x . Also, note that

$$Lt([a_1, \dots, a_n]) = n \text{ if and only if } \neq 0.$$

The previous observations are summarized in the next theorem:

Theorem

We have:

$$([a_1, \dots, a_n])_i = \begin{cases} a_i & \text{if } 1 \leq i \leq n, \\ 0 & \text{otherwise} \end{cases},$$

and

$$[(x)_1, \dots, (x)_n] = x \text{ if } n \geq Lt(x).$$