# THEORY OF COMPUTATION
## Universality - 9

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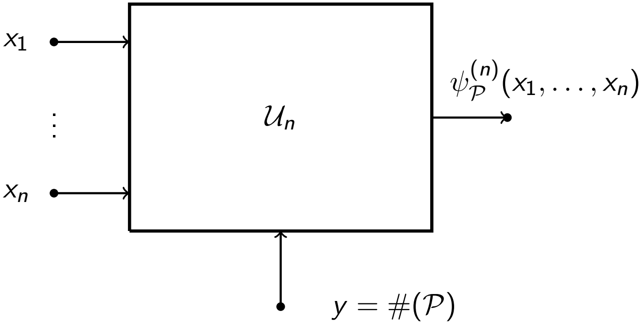
**Universality Theorem:**

### Theorem

*For each $n > 0$ there exists a partially computable function $\Phi^{(n)}$ such that if $\mathcal{P}$ is an $\mathcal{S}$-program with $\#(\mathcal{P}) = y$, then we have:*

$$\Phi^{(n)}(x_1, \ldots, x_n, y) = \psi_{\mathcal{P}}^{(n)}(x_1, \ldots, x_n)$$




Universal program $\mathcal{U}_n$ acts like an interpreter for computable functions of $n$ arguments.

The proof of the theorem consists in the construction of a program $\mathcal{U}_n$ for each $n > 0$ such that

$$\psi_{\mathcal{U}_n}^{(n+1)}(x_1, \dots, x_n, x_{n+1}) = \Phi^{(n)}(x_1, \dots, x_n, x_{n+1}),$$

when $x_{n+1}$ is the code of a program that computes $\Phi^{(n)}$.
The program $\mathcal{U}_n$ is called *universal*. It must

- keep track of the current snapshot of $\mathcal{P}$, and
- by decoding the number of the program being interpreted decide what to do next and do it.

Encoding the state of program $\mathcal{P}$ in a variable $S$:
If the $i^{\text{th}}$ variable in the list of variables has the value $a_i$ and all variables after the $m^{\text{th}}$ variables have value 0, the encoding of the state is $[a_1, \ldots, a_m]$.

**Example**

The state $Y = 0, X_1 = 2, X_2 = 1$ is encoded as

$$[0, 2, 0, 1] = 3^2 \cdot 7 = 63.$$

Note that the input variables occupy even numbered positions in the list of variables.

The variable $K$ contains the number that indicates the number of the instruction about to be executed.
Recall that the program $\mathcal{U}_n$ will compute

$$Y = \Phi^{(n)}(X_1, \ldots, X_n, X_{n+1}),$$

where $X_{n+1} = \#(\mathcal{P})$. The beginning of $\mathcal{U}$ consists of

$$Z \leftarrow X_{n+1} + 1$$
$$S \leftarrow \prod_{i=1}^{n} (p_{2i})^{X_i}$$
$$K \leftarrow 1$$

Note: the successive fragments of the program $\mathcal{U}_n$ will be shown in this color.

- If $X_{n+1} = \#(\mathcal{P})$, where $\mathcal{P}$ consists of instructions $I_1, \ldots, I_m$, then $Z$ gets the value $[\#(I_1), \ldots, \#(I_m)]$.
- $S$ is initialized at $[0, X_1, 0, X_2, \ldots, 0, X_n]$ which initializes the input variables and sets all other variables to 0.
- $K$ is given the initial value 1 so that the computation can begin with the first instruction.

Next, the line

$$[C] \quad \text{IF } K = \text{Lt}(Z) + 1 \vee (K = 0) \text{ GOTO } F$$

has the role of determine the end of the computation.

The current instruction must be decoded and executed:

$$U \leftarrow r((Z)_K)$$
$$P \leftarrow p_{r(U)+1}$$

Note that $(Z)_K = \langle a, \langle b, c \rangle\rangle$ is the number of the $K^{\text{th}}$ instruction. Thus, $U = \langle b, c \rangle$ is the code of the statement about to be executed.

The variable mentioned in the $K^{\text{th}}$ instruction is the $c + 1^{\text{st}}$ in the list, that is, $r(U) + 1^{\text{st}}$ in the list. Its current value is stored as the exponent to which $P$ divides $S$.

Depending on $b = \ell(U)$ and on the value of $\sim (P|S)$ we continue to certain labels:

$$\begin{array}{l} \text{IF } \ell(U) = 0 \text{ GOTO } N \\ \text{IF } \ell(U) = 1 \text{ GOTO } A \\ \text{IF } \sim (P|S) \text{ GOTO } N \\ \text{IF } \ell(U) = 2 \text{ GOTO } M \end{array}$$

- If $\ell(U) = 0$ the instruction is $V \leftarrow V$ and the computation does nothing to $S$.
- If $\ell(U) = 1$ the instruction is $V \leftarrow V + 1$, so 1 has to be added to the exponent of $P$ in the prime power factorization of $S$. Then, the computation executes a GOTO $A$.
- If $\ell(U)$ is neither 0 nor 1, then the current instruction is either $V \leftarrow V - 1$ or IF $V \neq 0$ GOTO $L$. In either case, if $P$ is not a divisor of $S$, that is, if the current value of $V$ is 0, the computation does nothing to $S$.
- If $P|S$ and $\ell(U) = 2$, the computation executes a GOTO $M$ ($M$ for minus), so 1 is subtracted from the exponent to which $P$ divides $S$.

The program continues with

$$K \leftarrow \min_{i \leqslant Lt(Z)}[\ell((Z)_i) + 2 = \ell(U)]$$
$$\text{GOTO } C$$

If $\ell(U) > 2$ and $P|S$ the current instruction is

$$\text{IF } V \neq 0 \text{ GOTO } L,$$

where $V$ has a non-zero value and $L$ is the label whose position is $\ell(U) - 2$. The instruction executed next is the first with this label, so $K$ should be the least $i$ such that $\ell((Z)_i) = \ell(U) - 2$. If there is no instruction with the appropriate label, $K$ gets 0, so the program terminates.

In either case, GOTO $C$ causes a jump to the beginning of the loop for the next instruction (if any) to be processed.

Next, we have:

$$\begin{array}{ll} [M] & S \leftarrow \lfloor S/P \rfloor \\ & \text{GOTO } N \\ [A] & S \leftarrow S \cdot P \\ [N] & K \leftarrow K + 1 \\ & \text{GOTO } C \end{array}$$

GOTO $C$ causes a jump to the beginning of the loop for the next instruction to be processed.

- $S \leftarrow \lfloor S/P \rfloor$ subtracts 1 from the value of the variable mentioned in the current instruction.
- $S \leftarrow S \cdot P$ adds 1 to the value of the variable mentioned in the current instruction.

The program concludes with

$$[F] \quad Y \leftarrow (S)_1$$

# The Program $\mathcal{U}_n$

| | |
|---|---|
| | $Z \leftarrow X_{n+1} + 1$ |
| | $S \leftarrow \prod_{i=1}^{n} (p_{2i})^{X_i}$ |
| | $K \leftarrow 1$ |
| $[C]$ | IF $K = \mathrm{Lt}(Z) + 1 \vee (K = 0)$ GOTO $F$ |
| | $U \leftarrow r((Z)_K)$ |
| | $P \leftarrow p_{r(U)+1}$ |
| | IF $\ell(U) = 0$ GOTO $N$ |
| | IF $\ell(U) = 1$ GOTO $A$ |
| | IF $\sim (P \mid S)$ GOTO $N$ |
| | IF $\ell(U) = 2$ GOTO $M$ |
| | $K \leftarrow \min_{i \leqslant Lt(Z)} [\ell((Z)_i) + 2 = \ell(U)]$ |
| | GOTO $C$ |
| $[M]$ | $S \leftarrow \lfloor S/P \rfloor$ |
| | GOTO $N$ |
| $[A]$ | $S \leftarrow S \cdot P$ |
| $[N]$ | $K \leftarrow K + 1$ |
| | GOTO $C$ |
| $[F]$ | $Y \leftarrow (S)_1$ |

On termination, the value of $Y$ is stored as the exponent on $p_1$ (which is 2).

For $n > 0$, the sequence

$$\Phi^{(n)}(x_1, \ldots, x_n, 0), \Phi^{(n)}(x_1, \ldots, x_n, 1), \ldots$$

enumerates all partially computable functions of $n$ variables.
An alternative notation is

$$\Phi_y^{(n)}(x_1, \ldots, x_n) = \Phi^{(n)}(x_1, \ldots, x_n, y).$$

For $n = 1$ we use the simplified notation

$$\Phi_y(x) = \Phi(x, y) = \Phi^{(1)}(x, y).$$

## Theorem

*Let $STP^{(n)}$ be the predicate:*

$$STP^{(n)}(x_1, \ldots, x_n, y, t) = \begin{cases} 1 & \textit{if program number } y \textit{ halts} \\ & \textit{after } t \textit{ or fewer steps} \\ & \textit{on inputs } x_1, \ldots, x_n \\ 0 & \textit{otherwise.} \end{cases}$$

*For each $n > 0$, $STP^{(n)}$ is primitive recursive.*

Note that $\mathrm{STP}^{(n)}$ is TRUE if there is a computation of program $y$ of length not greater than $t$ beginning with inputs $x_1, \ldots, x_n$.

The proof operates on numeric versions of the notion of snapshot. If $z$ represents a state $\sigma$ of the program $y$, the number $\langle i, z \rangle$ represents the snapshot $(i, \sigma)$. Therefore, for a program $\mathcal{P}$ whose code is

$$y = \#(\mathcal{P}) = [\#(I_1), \#(I_2), \ldots, \#(I_n)] - 1,$$

the code of instruction $I_i$ is $(y+1)_i$.

# Proof cont'd

The following functions extract the components of the $i^{\text{th}}$ instruction of the program number $y$, namely, the label, the variable number, the instruction type, and the label to which the $i^{\text{th}}$ instruction is pointing:

$$
\begin{aligned}
\text{LABEL}(i, y) &= \ell((y+1)_i), \\
\text{VAR}(i, y) &= r(r((y+1)_i)) + 1, \\
\text{INSTR}(i, y) &= \ell(r((y+1)_i)), \\
\text{LABEL}'(i, y) &= \ell(r((y+1)_i)) \dot{-} 2.
\end{aligned}
$$

# Proof cont'd

## Proof.

Next, we define some predicates that indicate, for program $y$ and snapshot $x$, which kind of action is to be performed:
Recall that if $x$ is a snapshot, $\ell(x)$ is the number of the instruction about to be executed and $r(x)$ represents the state of the program.

$$\begin{aligned} \text{SKIP}(x,y) \quad \Leftrightarrow \quad & [\text{INSTR}(\ell(x),y) = 0 \& \ell(x) \leqslant \text{Lt}(y+1)] \\ & \vee[\text{INSTR}(\ell(x),y) \geqslant 2 \& \sim (p_{\text{VAR}(\ell(x),y)} | r(x))] \end{aligned}$$

This says that if the type of the instruction is $V \leftarrow V$ or the instruction is an IF $V \neq 0$ GOTO $L$ and the value of $V$ is 0 (expressed as $\sim (p_{\text{VAR}(\ell(x),y)} | r(x))$), then the program skips to the next instruction. □

# Proof cont'd

## Proof.

$$\text{INCR}(x,y) \Leftrightarrow INSTR(\ell(x),y) = 1$$
$$\text{DECR}(x,y) \Leftrightarrow INSTR(\ell(x),y) = 2 \,\&\, p_{\text{VAR}(\ell(x),y)} | r(x)$$

INCR$(x,y)$ is TRUE if the instruction is $V \leftarrow V+1$; DECR$(x,y)$ is TRUE if the instruction is $V \leftarrow V-1$ and the value of $V$ is not 0; □

# Proof cont'd

## Proof.

$$\begin{aligned} \mathrm{BRANCH}(x,y) \quad \Leftrightarrow \quad & \mathrm{INSTR}(\ell(x),y) > 2 \& p_{\mathrm{VAR}(\ell(x),y)} | r(x) \\ & \&(\exists i)_{\leqslant \mathrm{Lt}(y+1)} \mathrm{LABEL}(i,y) = \mathrm{LABEL}'(\ell(x),y). \end{aligned}$$

BRANCH is TRUE if the instruction is of type IF $V \neq 0$ GOTO $L$, the value of the variable $V$ is not 0 (expressed by $p_{\mathrm{VAR}(\ell(x),y)} | r(x)$), and there exists an instruction with the label $L$, where the flow may continue. □

The function SUCC$(x, y)$ gives the representation of the successor of the snapshot represented by $x$ for the program $y$. This is a primitive recursive function defined by cases:

$$\text{SUCC}(x,y) = \begin{cases} \langle \ell(x)+1, r(x)\rangle & \text{if SKIP}(x,y), \\ \langle \ell(x)+1, r(x)\cdot p_{\text{VAR}(\ell(x),y)}\rangle & \text{if INCR}(x,y), \\ \langle \ell(x)+1, \lfloor r(x)/p_{\text{VAR}(\ell(x),y)}\rfloor\rangle & \text{if DECR}(x,y), \\ \langle \min_{i\leqslant \text{Lt}(y+1)}[\text{LABEL}(i,y) = \mathit{LABEL}'(\ell(x),y)] & \text{if } \mathit{BRANCH}(x,y) \\ \langle \text{Lt}(y+1)+1, r(x)\rangle & \text{otherwise.} \end{cases}$$

The function

$$\mathrm{INIT}^{(n)}(x_1, \ldots, x_n) = \langle 1, \prod_{i=1}^{n} (p_{2i})^{x_i} \rangle$$

gives the representation of the initial snapshot for inputs $x_1, \ldots, x_n$, and the predicate TERM given by

$$\mathrm{TERM}(x, y) \Leftrightarrow \ell(x) > Lt(y + 1)$$

tests whether $x$ represents a terminal snapshot for program $y$.

The function SNAP gives the numbers of successive snapshots produced by a program $y$. This function is primitive recursive because

$$\begin{aligned} \mathrm{SNAP}^{(n)}(x_1, \ldots, x_n, y, 0) &= \mathrm{INIT}^{(n)}(x_1, \ldots, x_n) \\ \mathrm{SNAP}^{(n)}(x_1, \ldots, x_n, y, i+1) &= \mathrm{SUCC}(\mathrm{SNAP}^{(n)}(x_1, \ldots, x_n, y, i), y). \end{aligned}$$

Thus,

$$\mathrm{STP}^{(n)}(x_1, \ldots, x_n, y, t) \Leftrightarrow \mathrm{TERM}(\mathrm{SNAP}^{(n)}(x_1, \ldots, x_n, y, t), y),$$

hence $\mathrm{STP}^{(n)}$ is primitive recursive.

An important consequence is the next theorem known as the **Normal Form Theorem**:

### Theorem

*Let $f$ be a partially computable function. Then, there is a primitive recursive predicate $R(x_1, \ldots, x_n, y)$ such that*

$$f(x_1, \ldots, x_n) = \ell\left(\min_z R(x_1, \ldots, x_n, z)\right).$$

## Proof.

Let $y_0$ be the number of a program that computes $f(x_1, \ldots, x_n)$. Let $R(x_1, \ldots, x_n, z)$ be the predicate defined by

$$\begin{aligned} R(x_1, \ldots, x_n, z) \quad \Leftrightarrow \quad & \mathrm{STP}^{(n)}(x_1, \ldots, x_n, y_0, r(z)) \\ & \&(r(\mathrm{SNAP}^{(n)}(x_1, \ldots, x_n, y_0, r(z))))_1 = \ell(z). \end{aligned}$$

Suppose that the right side of the above equality is defined. This means that there exists a number $z$ such that the computation of the program with number $y_0$ has reached a terminal snapshot in $r(z)$ or fewer steps and $\ell(z)$ is the value held in the output variable $Y$, that is, $\ell(z) = f(x_1, \ldots, x_n)$.
If the right side is undefined it must be the case that $\mathrm{STP}^{(n)}(x_1, \ldots, x_n, y_0, t)$ is false for all values of $t$, that is $f(x_1, \ldots, x_n) \uparrow$. □

A characterization of partially computable functions:

**Theorem**

*A function is partially computable if and only if it can be obtained from the initial functions by a finite number of applications of composition, recursion, and minimalization.*

**Proof.**

Every function that can be obtained by a finite number of applications of composition, recursion, and minimalization is clearly partially computable by previous results. □

# Proof cont'd

**Proof.**

Conversely, by the Normal Form Theorem, we can write any partially computable function as

$$f(x_1, \ldots, x_n) = \ell \left( \min_z R(x_1, \ldots, x_n, z) \right),$$

where $R$ is a primitive recursive predicate. Then $R$ is obtained from initial functions by a finite number of applications of composition and recursion. Finally, the given function is obtained from $R$ by one use of minimalization and then by application of $\ell$. □