Homework 4

Posted: November 6, 2024
Due: November 25, 2024

1. Define the predicate $P_k(x)$ as

$$P_k(x) = \begin{cases} 1 & \text{if } \Phi_x(x) = k, \\ 0 & \text{otherwise.} \end{cases}$$

   Prove that P_k is not computable.

   Hint: suppose that P_k were computable for any k. Then P_1 would also be computable.

2. Let A, B be two subsets of $\mathbb{N}$. Define the sets $A \oplus B$ and $A \otimes B$ as

$$\begin{aligned} A \oplus B &= \{2x \mid x \in A\} \cup \{2x + 1 \mid x \in B\} \\ A \otimes B &= \{\langle x, y \rangle \mid x \in A \text{ and } y \in B\}. \end{aligned}$$

   Prove that:

   (a) $A \oplus B$ is recursive if and only if A and B are both recursive;

   (b) if A and B are non-empty, then $A \otimes B$ is recursive if and only if A and B are both recursive.

   Hint: Express P_A and P_B using $P_{A \oplus B}$.

3. Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be a unary function. Prove that f is computable if and only if the set $S = \{2^x \cdot 3^{f(x)} \mid x \in \mathrm{Dom}(f)\}$ is recursively enumerable.

   Hint: Note that $S = \{[x, f(x)] \mid x \in \mathrm{Dom}(f)\}$, hence S is the set of values of a computable function.

4. If $A \leqslant_m B$, prove that $\overline{A} \leqslant_m \overline{B}$. Here $\overline{C}$ is the complement of C, that is, $\overline{C} = \mathbb{N} - C$.

5. Prove that the set $A = \{x \mid \mathrm{Dom}(\Phi_x) \neq \emptyset\}$ is recursively enumerable, but not recursive.