Problem Session 5

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Problem 1

Prove that the language $L = \{a^p \mid p \text{ is a prime }\}$ is not context-free.

Solution 1

Assume that L is context free and let n_L be the number defined in the Pumping Lemma. If p is a prime number such that $p \geqslant n_L$ and $w = a^p = xyzut \in L$ we have $|y| = a^k$, $|u| = a^\ell$ with $k + \ell \geqslant 1$, and $xy^{1+p}zu^{1+p}t \in L$, that is, $a^{p+kp+\ell p} \in L$, that is, $a^{p(1+k+\ell)} \in L$. Since $p(1 + k + \ell)$ is not a prime, this leads to a contradiction.

Problem 2

Prove that the language $L = \{a^n b^m \mid n \leqslant m^2\}$ is not context-free.

Solution 2

Assume that L is context-free and let $w = a^{m^2} b^m \in L$. If $m^2 + m \geqslant n_L$, where n_L is defined by the Pumping Lemma, we can pump the word $a^{m^2} b^m = xyzut$ with $|yzu| \leqslant n_L$ and $|yu| \geqslant 1$. Note that neither y nor u can straddle across the a-part and the b-part of w because pumping with create more than one alternance of a and b thus violating the definition of L. Therefore, it is the case that y is either in a^{m^2} or in b^m and the same is true for u.

Solution 2 (cont'd)

We need to distinguish three cases:

`Case I:` y occurs in a^{m^2} and u occurs in b^m, so $y = a^k$ and $u = b^\ell$ with $1 \leqslant k + \ell \leqslant m$ (because $|yzu| = k + |z| + \ell \leqslant n_L$ and $|yu| \geqslant 1$).

Suppose that $\ell \geqslant 1$. The Pumping Lemma implies $xy^0zu^0t \in L$. Therefore, $a^{m^2-k}b^{m-\ell} \in L$ and this implies $m^2 - k \leqslant (m-\ell)^2$. This leads to a contradiction because

$$\begin{aligned}(m-\ell)^2 &\leqslant (m-1)^2 \\ &\quad \text{(because } \ell \geqslant 1) \\ &= m^2 - 2m + 1 \\ &< m^2 - k \\ &\quad \text{(since } k \leqslant m).\end{aligned}$$

Suppose now that $\ell = 0$. Since $k + \ell \geqslant 1$, we have $k \geqslant 1$. The pumping Lemma implies $xy^2zu^2t \in L$. Therefore, $a^{m^2+k}b^m \in L$, which is impossible because $m^2 + k > m^2$.

Solution 2(cont'd)

`Case II`: If both y and u occur within a^{m^2} then pumping both y and u results in a word of the form $a^{m^2+k}b^m$ that is not in L.
`Case III`: If both y and u occur within b^m by pumping down both y and u results in a word of the form $a^{m^2}b^m - k$ with $k \geqslant 2$ that is not in L.

Problem 3

Prove that the language $L = \{a^n b^m c^{mn} \mid n, m \in \mathbb{N}\}$ is not context-free.

Solution 3

Suppose that L is context-free. By applying the Pumping Lemma to the string $a^m b^m c^{m^2} \in L$, where $m = n_L$ we can factor $a^m b^m c^{m^2} = xyzut$ with $|yzu| \leqslant n_L$ and $|yu| \leqslant 1$.

Note that y cannot span simultaneously both a^m and b^m because, if this would be the case, by pumping y we would obtain a string in which bs precede as. Similarly, u cannot span both b^m and c^{m^2}.

Therefore, y is contained in one of the regions a^m, b^m, or c^{m^2} and the same holds for u.

If y occurs in b^m and u occurs in c^{m^2} we have $y = b^k$ and $u = c^\ell$ with $1 \leqslant k + \ell \leqslant m$ because $|yzu| \leqslant m$ and $|yu| \geqslant 1$.

Solution 3(cont'd)

Suppose now that $k \geqslant 1$. Since $k + \ell \leqslant m$, it follows that $\ell < m$. The Pumping Lemma implies $xy^0zu^0t \in K$, hence $a^m b^{m-k} c^{m^2-\ell} \in L$, which implies $m(m-k) = m^2 - \ell$. This is impossible because $m(m-k) = m^2 - mk \leqslant m^2 - m < m^2 - \ell$ since $k \geq q$ and $\ell < m$.
Consider now the case when $k = 0$. Since $k + \ell \leqslant 1$, we have $\ell \geqslant 1$. From the Pumping Lemma we have $xy^0zu^0t \in L$, hence $a^m b^m c^{m^2-\ell} \in L$, which is impossible because $m \cdot m \neq m^2 - \ell$.
The remaining case can be dealt with using the pumping down.

Problem 4

Prove that the language
$H = \{a^i b^j c^k \mid i, j, k \in \mathbb{N} \text{ and } k = \max\{i, j\}\}$ is not context-free.

Solution 4

Suppose that H were context-free. Then, by the Pumping Lemma there is $n_H \in \mathbb{N}$ such that if $w = a^i b^j c^k \in H$ and $|w| \geq n_H$, then $w = xyzut$ such that $|y| \geq 1$ or $|u| \geq 1$, $|yzu| \leq n_H$ and $xy^n zu^n t \in H$ for $n \in \mathbb{N}$.
Choose $i = j = n_H$. If c does not occur in yu pumping will increase the number of as or bs without affecting the number of cs and we would obtain a contradiction. Therefore, c must occur in yzu.
Since $|yzu| \leqslant n_H$, it follows that the word yzu cannot contain an a.
By the Pumping Lemma the word xzt contains a number of n_H as and there are fewer than n_H symbols b or c in xzt. Thus, $xzt \notin H$.

Problem 5

Prove that the language $L = \{a^n ba^n ba^n \mid n \geqslant 1\}$ is not context-free.

Solution 5

Suppose that L were context-free and let n_L the constant defined by the Pumping Lemma. If $w = a^{n_L} b a^{n_L} b a^{n_L}$ we can factor w as $w = xyzut$ such that $|yzu| \leqslant n_L$ and $|yu| \geqslant 1$.
Note that yzu can be an infix of a^{n_L} or may contain one single b because the two bs are separated by n_L symbols.
In the first case, by pumping the as we break the balance between the three subwords that contain as. In the second case, if b occurs in y or u, then the number of bs will increase and this would violate the definition of L.
Note that we cannot have one single b symbol in yzu because this would imply that the as preceeding this b occur in the first a region and the as that follow this b will occur in the second a region. By pumping y and u we would grow the first two a regions which will become larger that the third a region.