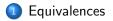
Finite Automata and Regular Languages (Preliminaries)

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Definition of Equivalences

Definition

An equivalence on a set S is a relation $\rho \subseteq S \times S$ that satisfies the following conditions:

- **Reflexivity:** $(x, x) \in \rho$ for every $x \in S$;
- Symmetry: $(x, y) \in \rho$ if and only if $(y, x) \in \rho$;
- **Transitivity:** if $(x, y) \in \rho$ and $(y, z) \in \rho$ then $(x, z) \in \rho$.

Define the relation ρ_m on $\mathbb N$ as

$$ho_{\textit{m}} = \{(\textit{p},\textit{q}) \mid \textit{p},\textit{q} \in \mathbb{N} \text{ and } \textit{p} - \textit{q} = \textit{km} \text{ for some } \textit{k} \in \mathbb{Z}\}.$$

- $(p, p) \in \rho_m$ because $p p = 0 \cdot m$, so ρ_m is reflexive;
- if $(p,q) \in \rho_m$, then p-q = km, hence q-p = (-k)m, which means that $(q,p) \in \rho_m$;
- if $(p,q) \in \rho_m$ and $(q,r) \in \rho_m$ then p-q = km and q-r = hm; therefore, p-r = p-q+q-r = (k+h)m, so $(p,r) \in \rho_m$, hence ρ_m is transitive.

Thus, ρ_m is an equivalence relation on \mathbb{N} .

Let L be the set of lines in a plane Π . Define $\ell \parallel \ell'$ if ℓ is parallel to ℓ' .

- we have $\ell \parallel \ell$, so \parallel is reflexive;
- if $\ell \parallel \ell'$, then $\ell' \parallel \ell$ hence \parallel is symmetric;
- if $\ell \parallel \ell'$ and $\ell' \parallel \ell''$, then $\ell \parallel \ell''$, so parallel is transitive.

Thus, " \parallel " is an equivalence relation on the set *L*.

Definition

Let S be a set and let ρ be an equivalence on S.

The ρ -equivalence class of an element x of S is the set [x] defined by

$$[x] = \{z \in S \mid (x, z) \in \rho\}.$$

The quotient set S/ρ is the set of all equivalence classes defined by ρ on the set S.

Note that:

- we have x ∈ [x] because (x, x) ∈ ρ, so none of the equivalence classes is empty;
- if $y \in [x]$, then $x \in [y]$ because ρ is symmetric; thus, in this case, [y] = [x].
- if two equivalence classes are distinct, they are disjoint.

Suppose that $[y] \neq [x]$ and there exists $t \in [x] \cap [y]$. Then $(x, t) \in \rho$ and $(y, t) \in \rho$, which means that $(t, y) \in \rho$. By transitivity $(x, y) \in \rho$, which implies [x] = [y], contradicting the initial assumption.

Definition

Let S be a non-empty set. A partition on S is a collection of sets $\pi = \{B_i \mid B_i \subseteq S \text{ for } i \in I\}$ such that

•
$$B_i \neq \emptyset$$
 for all $i \in I$;

•
$$B_i \cap B_j = \emptyset$$
 for $i, j \in I$ and $i \neq j$;

• $\bigcup_{i\in I} B_i = S.$

If ρ is an equivalence on a set S, its set of classes

$\{[x] \mid x \in S\}$

is a partition of the set S. This shows how we can "move" from equivalences to partitions.

Conversely, if $\pi = \{B_i \mid B_i \subseteq S \text{ for } i \in I\}$ is a partition of S, an equivalence ρ_{π} is defined by

 $(x, y) \in \rho_{\pi}$ if and only if there is a block B of π such that $\{x, y\} \subseteq B$.

Definition

Let π be a partition of a set *S*. A subset *U* of *S* is π -saturated if it equals the union of a collection of blocks of π .

Let $S = \{1, 2, \dots, 9\}$ be a set and let

$$\pi = \{\{1, 2, 7\}, \{4, 6\}, \{3, 5, 8, 9\}\}.$$

The set S has $2^9=512$ subsets. The following 8 subsets of S are $\pi\text{-saturated:}$