Finite Automata and Regular Languages (part II)

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1. Nondeterministic Automata

Definition

A nondeterministic finite automaton (ndfa) is a quintuple $\mathcal{M} = (A, Q, \delta, q_0, F)$, where A is the input alphabet of $\mathcal{M}$, Q is a finite set of states, $\delta : Q \times A \longrightarrow \mathcal{P}(Q)$ is the transition function, $q_0 \in Q$ is the initial state, and $F \subseteq Q$ is the set of final states of $\mathcal{M}$. We assume $A \cap Q = \emptyset$.

Example

Consider the ndfa

$$\mathcal{M} = (\{a, b\}, \{q_0, q_1, q_2, q_3, q_4\}, \delta, q_0, \{q_1, q_3\}),$$

whose transition function is defined by the table:

Input	State				
	q_0	q_1	q_2	q_3	q_4
a	$\{q_1, q_2\}$	$\emptyset$	$\{q_3\}$	$\emptyset$	$\emptyset$
b	$\{q_0\}$	$\{q_3\}$	$\{q_4\}$	$\emptyset$	$\emptyset$

Note the presence of pairs (q, a) such that $\delta(q, a) = \emptyset$. We refer to such pairs as blocking situations of $\mathcal{M}$. For instance, (q_1, a) is a blocking situation of $\mathcal{M}$.

Extending the transition function for an ndfa

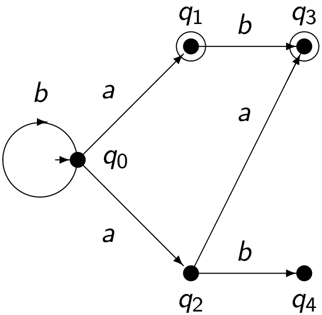
As in the case of the dfa, we can extend the ndfa's transition function δ, defined on single characters, to δ^*, defined on words.
Starting from the transition function δ, we define the function $\delta^* : Q \times A^* \longrightarrow \mathcal{P}(Q)$ as follows:

$$\begin{aligned} \delta^*(q, \lambda) &= \{q\} \\ \delta^*(q, xa) &= \bigcup_{q' \in \delta^*(q,x)} \delta(q', a) \end{aligned}$$

Graphs of ndfas

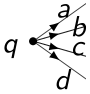
- If $\mathcal{M} = (A, Q, \delta, q_0, F)$ is an ndfa, then the graph of $\mathcal{M}$ is the marked, directed multigraph $\mathcal{G}(\mathcal{M})$, whose set of vertices is the set of states Q.
- The set of edges of $\mathcal{G}(\mathcal{M})$ consists of all pairs (q, q') such that $q' \in \delta(q, a)$ for some $a \in A$; an edge (q, q') is labeled by the input symbol a, where $q' \in \delta(q, a)$.
- The initial state q_0 is denoted by an incoming arrow with no source, and the final states are circled.

If $q' \in \delta^*(q, x)$, then there is a path in the graph $\mathcal{G}(\mathcal{M})$ labeled by x that leads from q to q'.

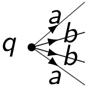

q_1
b
q_3
b
a
a
q_0
a
b
q_2
q_4

Comparing dfas and ndfas

- In the graph of a dfa $\mathcal{M} = (A, Q, \delta, q_0, F)$ you must have exactly one edge emerging from every state q and for every input symbol $a \in A$.



- In the graph of an ndfa $\mathcal{M} = (A, Q, \delta, q_0, F)$ you may have states where no arrow emerges, or states where several arrow labeled with the same symbol emerge. Also, not every symbol needs to appear as a label of an emerging edge.



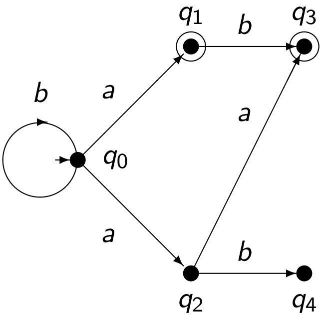
Definition

The language accepted by the ndfa $\mathcal{M} = (A, Q, \delta, q_0, F)$ is

$$L(\mathcal{M}) = \{x \in A^* \mid \delta^*(q_0, x) \cap F \neq \emptyset\}.$$

In other words, $x \in L(\mathcal{M})$ if there exists a path in the graph of $\mathcal{M}$ labeled by x that leads from the initial state into one of the final states. Note that it is not necessary that all paths labeled by x lead to a final state; the existence of one such path suffices to put x into the language $L(\mathcal{M})$.

Example



Note that $ab \in L(\mathcal{M})$ because of the existence of the path (q_0, q_1, q_3) labeled by this word and the fact that q_3 is a final state. On the other hand, (q_0, q_2, q_4) is another path labeled by x but $q_4 \notin F$.

Example (cont'd)

This ndfa is simple enough to allow an easy identification of all types of words in $L(\mathcal{M})$:

1. The final state q_1 can be reached by applying an arbitrary number of bs followed by an a.
2. The final state q_3 can be reached by a path of the form $(q_0, \ldots, q_0, q_1, q_3)$, that is by a word of the form $b^k ab$ for $k \in \mathbb{N}$.
3. The same final state q_3 can be reached via q_2. Input words that allow this transition have the form $b^k aa$ for $k \in \mathbb{N}$.

Thus, we have

$$L(\mathcal{M}) = \{b\}^* a \cup \{b\}^* ab \cup \{b\}^* aa = \{b\}^* \{a, ab, aa\}.$$

Example

Consider an alphabet $A = \{a_0, \ldots, a_{n-1}\}$ and a binary relation $\rho \subseteq A \times A$. The language

$$L_\rho = \{a_{i_0} \cdots a_{i_p} \mid p \in \mathbb{N}, (a_{i_j}, a_{i_{j+1}}) \in \rho \text{ for } 0 \leq j \leq p-1\}$$

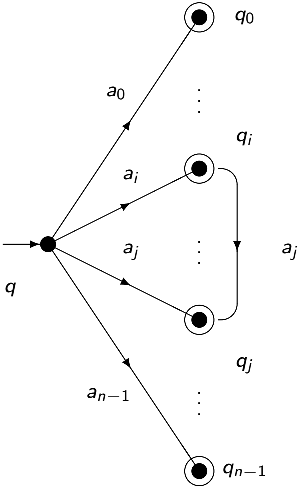
is accepted by the ndfa $\mathcal{M}_\rho = (A, Q, \delta, q, F)$, where $Q = \{q, q_0, \ldots, q_{n-1}\}$, $F = \{q_0, \ldots, q_{n-1}\}$, and δ is given by

- $\delta(q, a_i) = \{q_i\}$ for $0 \leq i \leq n-1$;
- for every i, j such that $0 \leq i, j \leq n-1$,

$$\delta(q_i, a_j) = \{q_j \in Q \mid (a_i, a_j) \in \rho\}.$$

Example (cont'd)

Note that if $(a_i, a_j) \notin \rho$, then (q_i, a_j) is a blocking situation. The existence of these blocking situations is precisely what makes this device a nondeterministic automaton.



We show that $L_\rho = L(\mathcal{M}_\rho)$. Let $x = a_{i_0} \cdots a_{i_p}$ be a word in L_ρ with $p \geq 0$. We prove by induction on $p = |x| - 1$ that $x \in L(\mathcal{M}_\rho)$ and that $\delta^*(q, x) = \{q_{i_p}\}$. The base case, $p = 0$, is immediate, since the condition $(a_{i_j}, a_{i_{j+1}}) \in \rho$ for $0 \leq j \leq p - 1$ is vacuous.
Suppose that the statement holds for words in L_ρ of length at most p and let $x = a_{i_0} \cdots a_{i_p}$ be a word in L_ρ of length $p + 1$. It is clear that the word $y = a_{i_0} \cdots a_{i_{p-1}}$ belongs to L_ρ. By the inductive hypothesis, $y \in L(\mathcal{M}_\rho)$ and $\delta^*(q, y) = \{q_{i_{p-1}}\}$. Since $(a_{i_{p-1}}, a_{i_p}) \in \rho$ (by the definition of L_ρ), we have $\delta(q_{i_{p-1}}, a_{i_p}) = \{q_{i_p}\}$, so

$$q_{i_p} \in \bigcup_{q' \in \delta(q,y)} \delta(q', a_{i_p}) = \delta^*(q, y a_{i_p}) = \delta^*(q, x).$$

Therefore, $x \in L(\mathcal{M}_\rho)$.

To prove the converse inclusion $L(\mathcal{M}_\rho) \subseteq L_\rho$ we use an argument by induction on $|x| \geq 1$, where x is a word from $L(\mathcal{M}_\rho)$, to show that if $x = a_{i_0} \cdots a_{i_p} \in L(\mathcal{M}_\rho)$, then $\delta^*(q, x) = \{q_{i_p}\}$ and $x \in L_\rho$. Again, the base case is immediate.

Suppose that the statement holds for words in $L(\mathcal{M}_\rho)$ of length less than $p + 1$ that belong to L_ρ, and let $x = a_{i_0} \cdots a_{i_p}$ be a word in $L(\mathcal{M}_\rho)$ of length $p + 1$. If $y = a_{i_0} \ldots a_{i_{p-1}}$, it is easy to see that $y \in L(\mathcal{M}_\rho)$ because no blocking situation may arise in $\mathcal{M}_\rho$ while the symbols of y are read. Therefore, by the inductive hypothesis, $y \in L_\rho$ and $\delta^*(q, y) = \{q_{i_{p-1}}\}$. Further, since $\delta(q_{i_{p-1}}, a_{i_p}) \neq \emptyset$, it follows that $(a_{i_{p-1}}, a_{i_p}) \in \rho$, so $x \in L_\rho$, and $\delta^*(q, x) = q_{i_p}$.

Thus, L_ρ is accepted by the ndfa $\mathcal{M}_\rho$.

Let $\mathcal{M} = (A, Q, \delta, q_0, F)$ be a nondeterministic automaton, and let $\Delta : \mathcal{P}(Q) \times A \longrightarrow \mathcal{P}(Q)$ be defined by

$$\Delta(S, a) = \bigcup_{q \in S} \delta(q, a) \tag{1}$$

for every $S \subseteq Q$ and $a \in A$. Starting from Δ, we define $\Delta^* : \mathcal{P}(Q) \times A^* \longrightarrow \mathcal{P}(Q)$ in the manner used for the transition functions of deterministic automata. Namely, we define

$$\Delta^*(S, \lambda) = S \tag{2}$$

$$\Delta^*(S, xa) = \Delta(\Delta^*(S, x), a) \tag{3}$$

for every $S \subseteq Q$ and $a \in A$.

Lemma

The functions Δ *and* Δ^* *defined above satisfy the following properties:*

1. *For every family of sets* $\{S_0, \ldots, S_{n-1}\}$ *and every* $a \in A$*, we have:*

$$\Delta\left(\bigcup_{0\le i\le n-1} S_i, a\right) = \bigcup_{0\le i\le n-1} \Delta(S_i, a).$$

2. *For every set* $S \subseteq Q$ *and* $x \in A^*$ *we have*

$$\Delta^*(S, x) = \bigcup_{q\in S} \delta^*(q, x).$$

Proof

The first part of the lemma is immediate, because

$$\begin{aligned}\Delta(\bigcup_{0\le i\le n-1} S_i, a) &= \bigcup\{\delta(q,a) \mid q \in \bigcup_{0\le i\le n-1} S_i\}\\ &= \bigcup_{0\le i\le n-1}\{\delta(q,a) \mid q \in S_i\}\\ &= \bigcup_{0\le i\le n-1} \Delta(S_i, a).\end{aligned}$$

The argument for the second part of the lemma is by induction on $|x|$. For the basis step, we have $|x| = 0$, so $x = \lambda$, and $\Delta^*(S, \lambda) = S$, $\bigcup_{q \in S} \delta^*(q, \lambda) = \bigcup_{q \in S} \{q\} = S$.
Suppose that the argument holds for words of length n, and let $x = za$ be a word of length $n + 1$. We have

$$\begin{aligned}
\Delta^*(S, x) &= \Delta^*(S, za) \\
&= \Delta(\Delta^*(S, z), a) \\
&= \Delta(\bigcup_{q \in S} \delta^*(q, z), a) (\text{by ind. hyp.}) \\
&= \bigcup_{q \in S} \Delta(\delta^*(q, z), a) \\
&= \bigcup_{q \in S} \bigcup_{r \in \delta^*(q,z)} \delta(r, a) = \bigcup_{q \in S} \delta^*(q, za) = \bigcup_{q \in S} \delta^*(q, x).
\end{aligned}$$

Nondeterministic automata can be regarded as generalizations of deterministic automata in the following sense. If $\mathcal{M} = (A, Q, \delta, q_0, F)$ is a deterministic automaton, consider a nondeterministic automaton $\mathcal{M}' = (A, Q, \delta', q_0, F)$, where $\delta'(q, a) = \{\delta(q, a)\}$. It is easy to verify that for every $q \in Q$ and $x \in A^*$ we have $\delta'^*(q, x) = \{\delta^*(q, x)\}$. Therefore,

$$\begin{aligned} L(\mathcal{M}') &= \{x \in A^* \mid \delta'^*(q_0, x) \cap F \neq \emptyset\} \\ &= \{x \in A^* \mid \{\delta^*(q_0, x)\} \cap F \neq \emptyset\} \\ &= \{x \in A^* \mid \delta^*(q_0, x) \in F\} \\ &= L(\mathcal{M}). \end{aligned}$$

In other words, for every deterministic finite automaton there exists a nondeterministic one that recognizes the same language.

Theorem

For every nondeterministic automaton, there exists a deterministic automaton that accepts the same language.

Proof

Let $\mathcal{M} = (A, Q, \delta, q_0, F)$ be a nondeterministic automaton. Define the function Δ as in the equality

$$\Delta(S, a) = \bigcup_{q \in S} \delta(q, a),$$

and consider the deterministic automaton $\mathcal{M}' = (A, \mathcal{P}(Q), \Delta, \{q_0\}, \{S \mid S \subseteq Q \text{ and } S \cap F \neq \emptyset\})$. For every $x \in A^*$ we have the following equivalent statements:

1. $x \in L(\mathcal{M})$;
2. $\delta^*(q_0, x) \cap F \neq \emptyset$;
3. $\Delta^*(\{q_0\}, x) \cap F \neq \emptyset$;
4. $x \in L(\mathcal{M}')$.

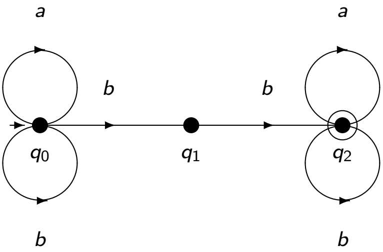
This proves that $L(\mathcal{M}) = L(\mathcal{M}')$.

Example

Consider the nondeterministic finite automaton

$$\mathcal{M} = (\{a, b\}, \{q_0, q_1, q_2\}, \delta, q_0, \{q_2\})$$

whose graph is given below.



It is easy to see that the language accepted by this automaton is $A^* bbA^*$, that is the language that consists of all words that contain two consecutive b symbols.

The graph of the nondeterministic automaton is simpler than the graph of the previous deterministic automaton; this simplification is made possible by the nondeterminism.

Graph of the Equivalent ndfa