Finite Automata and Regular Languages (part III)

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1. Transition Systems

Nondeterministic finite automata can be further generalized by allowing transitions between states without reading any input symbol.

Definition

A transition system (ts) is a 5-tuple $\mathcal{T} = (A, Q, \theta, Q_0, F)$, where A, Q, and F are as in a finite automaton, θ is a finite relation, $\theta \subseteq Q \times A^* \times Q$, called the transition relation of $\mathcal{T}$, and Q_0 is a nonempty subset of Q called the set of initial states, and F is the set of final states.
A transition in $\mathcal{T}$ is a triple $(q, x, q') \in \theta$. We refer to transitions of the form (q, λ, q') as null transitions.

Transition systems are conveniently represented by labeled directed multigraphs. Namely, if $\mathcal{T} = (A, Q, \theta, Q_0, F)$ is a transition system, then its graph is a labeled directed multigraph $G(\mathcal{T})$.

- $G(\mathcal{T})$ has Q as its set of vertices;
- each directed edge e from q to q' labelled x corresponds to a triple $(q, x, q') \in \theta$, and every such triple is represented by an edge in G.

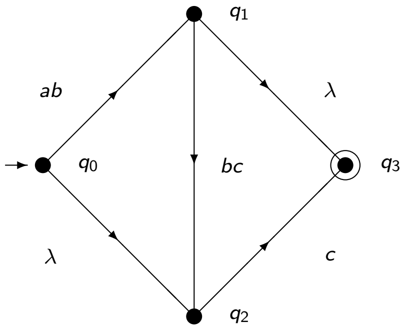
Unlike the graph of a dfa or an ndfa, the edges of the directed graph of a transition system can be labelled by words, including the null word.

Example

The graph of the transition system

$$\mathcal{T}_1 = (\{a, b, c\}, \{q_0, q_1, q_2, q_3\}, \theta, \{q_0\}, \{q_3\}),$$

where θ is given by
$\theta = \{(q_0, ab, q_1), (q_0, \lambda, q_2), (q_1, bc, q_2), (q_1, \lambda, q_3), (q_2, c, q_3)\}$ is shown below:



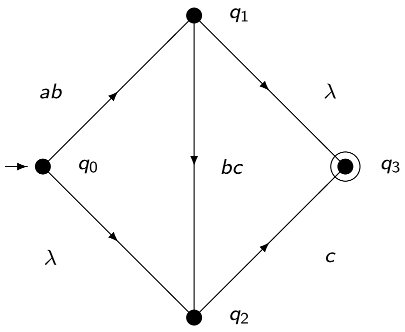
Extending the transition relation

As we did with the transition functions of dfas and ndfas, we wish to extend the transition relation θ of a transition system $\mathcal{T}$ to the set $Q \times A^* \times Q$. The extension $\theta^* \subseteq Q \times A^* \times Q$ is given next.

1. For every $q \in Q$ define $(q, \lambda, q) \in \theta^*$.
2. Every triple $(q, x, q') \in \theta$ belongs to θ^*.
3. If $(q, x, q'), (q', y, q'') \in \theta^*$, then $(q, xy, q'') \in \theta^*$.

Note that $(q, w, q') \in \theta^*$ if and only if there is a path in $G(\mathcal{T})$ that begins with q and ends with q' such that the concatenated labels of the directed edges of this path form the word w.

Example



If $\mathcal{T}$ is the above transition system, then $(q_0, abbcc, q_3) \in \theta^*$ because $(q_0, ab, q_1), (q_1, bc, q_2), (q_2, c, q_3) \in \theta$. Similarly, $(q_0, c, q_3) \in \theta^*$ because $(q_0, \lambda, q_2), (q_2, c, q_3) \in \theta$.

Definition

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system. The language accepted by $\mathcal{T}$ is

$$L(\mathcal{T}) = \{x \in A^* \mid (q_0, x, q) \in \theta^* \text{ for some } q_0 \in Q_0, q \in F\}.$$

Thus, a word x belongs to $L(\mathcal{T})$ if there is a path in $G(\mathcal{T})$ that begins in an initial state $q_0 \in Q_0$, labeled by x, such that the path ends in one of the states of F, the set of final states.

Transition systems generalize ndfas

If $\mathcal{M} = (A, Q, \delta, q_0, F)$ is a nondeterministic automaton, define the transition system $\mathcal{T}_{\mathcal{M}} = (A, Q, \theta, \{q_0\}, F)$, where

$$\theta = \{(q, a, q') \mid q, q' \in Q, a \in A \text{ and } q' \in \delta(q, a)\}.$$

It can be shown by induction on $|x|$ that $(q, x, q') \in \theta^*$ if and only if $q' \in \delta^*(q, x)$. This implies that $L(\mathcal{T}_{\mathcal{M}}) = L(\mathcal{M})$. Therefore, every regular language can be accepted by a transition system. Furthermore, any language that can be accepted by a transition system is regular.

Lemma

For every transition system $\mathcal{T} = (A, Q, \theta, Q_0, F)$ *there exists a transition system* $\mathcal{T}' = (A, Q', \theta', Q_0, F)$ *such that* $(q, x, q_1) \in \theta'$ *implies* $|x| \leq 1$ *and* $L(\mathcal{T}') = L(\mathcal{T})$.

Proof

Define the relation θ' and the set Q' as follows:

1. Every state $q \in Q$ also belongs to Q'.
2. Every triple $(q, x, q') \in \theta$ such that $|x| \leq 1$ also belongs to θ'.
3. If $t = (q, x, q') \in \theta$ such that $x = a_0 \ldots a_{n-1}$ and $n \geq 2$, add $n-1$ new states $q_0^t, \ldots q_{n-2}^t$ to Q' and the triples

$$(q, a_0, q_0^t), (q_0^t, a_1, q_1^t), \ldots, (q_{n-2}^t, a_{n-1}, q')$$

   to θ'.

The ts $\mathcal{T}'$ clearly satisfies the conditions of the lemma, since $(q, x, q') \in \theta^*$ if and only if $(q, x, q') \in \theta'^*$.

Theorem

For every transition system $\mathcal{T} = (A, Q, \theta, Q_0, F)$ *there exists a deterministic finite automaton* $\mathcal{M}$ *such that* $L(\mathcal{T}) = L(\mathcal{M})$.

Proof.

By the previous Lemma we can assume that $(q, x, q') \in \theta$ implies $|x| \leq 1$. Define the deterministic finite automaton $\mathcal{M} = (A, \mathcal{P}(Q), \Delta, Q_0', F')$, where the initial state of $\mathcal{M}$ is

$$Q_0' = \{q \in Q \mid (q_0, \lambda, q) \in \theta^* \text{ for some } q_0 \in Q_0\},$$

the set of final states is $F' = \{S \mid S \subseteq Q, S \cap F \neq \emptyset\}$, and the function Δ is defined by

$$\Delta(S, a) = \{q' \in Q \mid (q, a, q') \in \theta^* \text{ for some } q \in S\},$$

for every $S \subseteq Q$ and $a \in A$. □

Proof (cont'd)

It is not difficult to verify, by induction on $|x|$, that

$$\Delta^*(Q_0', x) = \{q' \in Q \mid (q_0, x, q') \in \theta^* \text{ for some } q_0 \in Q_0\},$$

for $x \in A^*$. For the basis case, $|x| = 0$, so the above equality becomes

$$Q_0' = \{q' \in Q \mid (q_0, \lambda, q') \in \theta^* \text{ for some } q_0 \in Q_0\},$$

which holds by the definition of Q_0'.

Proof (cont'd)

Suppose that the equality holds for words of length n, and let y be a word of length $n+1$. We can write $y = xa$, so

$$
\begin{aligned}
\Delta^*(Q_0', y) &= \Delta^*(Q_0', xa) \\
&= \Delta(\Delta^*(Q_0', x), a) \\
&= \Delta(\{q' \in Q \mid (q_0, x, q') \in \theta^* \text{ for some } q_0 \in Q_0\}, a) \\
&\quad \text{(by the inductive hypothesis)} \\
&= \{r \in Q \mid (q', a, r) \in \theta^*, \text{ for some } q' \text{ such that} \\
&\quad (q_0, x, q') \in \theta^* \text{ for some } q_0 \in Q_0\} \\
&\quad \text{(by the definition of } \Delta) \\
&= \{r \in Q \mid (q_0, xa, r) \in \theta^* \text{ for some } q_0 \in Q_0\} \\
&\quad \text{(by the definition of } \theta^*) \\
&= \{r \in Q \mid (q_0, y, r) \in \theta^* \text{ for some } q_0 \in Q_0\},
\end{aligned}
$$

which concludes our inductive argument.

Proof (cont'd)

From this it follows that $L(\mathcal{M}) = L(\mathcal{T})$. By definition, $x \in L(\mathcal{M})$ if and only if $\Delta^*(Q_0', x) \in F'$. This is equivalent to $\Delta^*(Q_0', x) \cap F \neq \emptyset$. This is equivalent to the existence of a state $q' \in F'$ such that $(q_0, x, q') \in \theta^*$ for some $q_0 \in Q_0$, and this is equivalent to $x \in L(\mathcal{T})$.

Corollary

The class of languages that are accepted by transition systems is the class $\mathcal{R}$ of regular languages.

Definition

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system. The λ-closure is the mapping $K_{\mathcal{T}} : \mathcal{P}(Q) \longrightarrow \mathcal{P}(Q)$ given by

$$K_{\mathcal{T}}(S) = \{q \in Q \mid (s, \lambda, q) \in \theta^* \text{ for some } s \in S\}.$$

The set $K_{\mathcal{T}}(S)$ comprises the states in S plus all the states that can be reached from a state in S using a series of λ-transitions.

Theorem

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system. The λ-closure of $\mathcal{T}$ has the following properties.

1. $S \subseteq K_{\mathcal{T}}(S)$;
2. $S \subseteq S'$ *implies* $K_{\mathcal{T}}(S) \subseteq K_{\mathcal{T}}(S')$;
3. $K_{\mathcal{T}}(K_{\mathcal{T}}(S)) = K_{\mathcal{T}}(S)$,

for every $S, S' \in \mathcal{P}(Q)$.

Proof

Since $(s, \lambda, s) \in \theta^*$ it is immediate that $S \subseteq K_{\mathcal{T}}(S)$ for every $S \in \mathcal{P}(Q)$. The second part of the theorem is a direct consequence of the definition of $K_{\mathcal{T}}$.
Note that Parts (i) and (ii) imply $K_{\mathcal{T}}(S) \subseteq K_{\mathcal{T}}(K_{\mathcal{T}}(S))$. Let $q \in K_{\mathcal{T}}(K_{\mathcal{T}}(S))$. There is a state $s \in S$ and a state $r \in K_{\mathcal{T}}(S)$ such that $(s, \lambda, r) \in \theta^*$ and $(r, \lambda, q) \in \theta^*$. By the definition of θ^* we obtain $(s, \lambda, q) \in \theta^*$, so $q \in K_{\mathcal{T}}(S)$. This implies $K_{\mathcal{T}}(K_{\mathcal{T}}(S)) \subseteq K_{\mathcal{T}}(S)$, which gives the last part of the theorem.

Definition

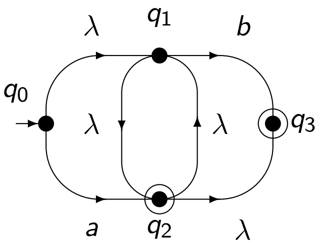
Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ be a transition system. A $K_{\mathcal{T}}$-closed subset of Q is a set S such that $S \subseteq Q$ and $K_{\mathcal{T}}(S) = S$.

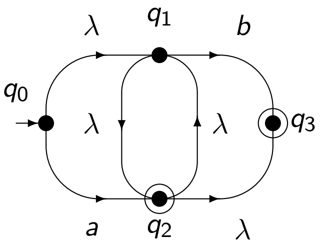
Example

Consider the transition system

$$\mathcal{T} = (\{a, b\}, \{q_0, q_1, q_2, q_3\}, \theta, \{q_0\}, \{q_2, q_3\})$$

whose graph is shown





S	$K_{\mathcal{T}}(S)$	S	$K_{\mathcal{T}}(S)$
$\emptyset$	$\emptyset$	$\{q_1, q_2\}$	$\{q_1, q_2, q_3\}$
$\{q_0\}$	Q	$\{q_1, q_3\}$	$\{q_1, q_2, q_3\}$
$\{q_1\}$	$\{q_1, q_2, q_3\}$	$\{q_2, q_3\}$	$\{q_1, q_2, q_3\}$
$\{q_2\}$	$\{q_1, q_2, q_3\}$	$\{q_0, q_1, q_2\}$	Q
$\{q_3\}$	$\{q_3\}$	$\{q_0, q_1, q_3\}$	Q
$\{q_0, q_1\}$	Q	$\{q_0, q_2, q_3\}$	Q
$\{q_0, q_2\}$	Q	$\{q_1, q_2, q_3\}$	$\{q_1, q_2, q_3\}$
$\{q_0, q_3\}$	Q	$\{q_0, q_1, q_2, q_3\}$	Q

The closed subsets of Q are $\emptyset$, $\{q_3\}$, $\{q_1, q_2, q_3\}$, and Q itself.

Theorem

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ *be a transition system and let* $\mathcal{M} = (A, \mathcal{P}(Q), \Delta, Q_0', F')$ *be the dfa constructed in earlier. The set* $\Delta(S, a)$ *is a* $K_{\mathcal{T}}$*-closed set of states for every subset* S *of* Q *and* $a \in A$.

Proof

To prove the theorem it suffices to show that $K_{\mathfrak{T}}(\Delta(S, a)) \subseteq \Delta(S, a)$. Let $p \in K_{\mathfrak{T}}(\Delta(S, a))$. There is $p_1 \in \Delta(S, a)$ such that $(p_1, \lambda, p) \in \theta^*$. The definition of $\Delta(S, a)$ implies the existence of $q \in S$ such that $(q, a, p_1) \in \theta^*$. Thus, $(q, a, p) \in \theta^*$, so $p \in \Delta(q, a)$.

Corollary

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ *be a transition system, and let* $\mathcal{M} = (A, \mathcal{P}(Q), \Delta, Q_0', F')$ *be the constructed dfa. The accessible states of the dfa* $\mathcal{M}$ *are* $K_{\mathcal{T}}$*-closed subsets of* Q*.*

Proof.

The initial state Q_0' of $\mathcal{M}$ is obviously closed. If Q' is an accessible state of $\mathcal{M}$, then $Q' = \Delta(S, a)$ for some $S \subseteq Q$. Therefore Q' is closed. □

Theorem

Let $\mathcal{T} = (A, Q, \theta, Q_0, F)$ *be a transition system, and let* $\mathcal{M} = (A, \mathcal{P}(Q), \Delta, Q_0', F')$ *be the dfa constructed earlier. Then,* $\Delta(S, a) = K_{\mathcal{T}}(\{q \in Q \mid (s, a, q) \in \theta \text{ for some } s \in S\})$*, where* S *is an accessible state of* $\mathcal{M}$*.*

Proof

Let $s \in S$. Note that if $(s, a, q) \in \theta$ and $(q, \lambda, q_1) \in \theta^*$, then by the definition of θ^* we have $(s, a, q_1) \in \theta^*$. Therefore, $K_{\mathfrak{T}}(\{q \in Q \mid (s, a, q) \in \theta\} \subseteq \Delta(S, a)$.
To prove the converse inclusion, $\Delta(S, a) \subseteq K_{\mathfrak{T}}(\{q \in Q \mid (s, a, q) \in \theta\}$, let $(s, a, q_1) \in \theta^*$. Then, there is a path in $G(\mathfrak{T})$ that begins with s and ends with q_1 such that the concatenated labels of the directed edges of this path form the word a. This implies the existence of the states $p, p' \in Q$ such that $(s, \lambda, p) \in \theta^*$, $(p, a, p_1) \in \theta$, and $(p_1, \lambda, q_1) \in \theta^*$. Since S is $K_{\mathfrak{T}}$-closed it follows that $p \in S$ and this gives the desired conclusion.

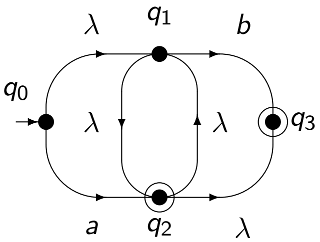
An Algorithm for Constructing a dfa corresponding to a ts

Input: A transition system $\mathcal{T} = (A, Q, \theta, Q_0, F)$.
Output: An accessible dfa $\mathcal{M}_1$ such that $L(\mathcal{M}_1) = L(\mathcal{T})$.
Method: Compute the increasing sequence of collections of subsets of Q, $\mathcal{Q}_0, \ldots, \mathcal{Q}_i, \ldots$, where

$$\begin{aligned} \mathcal{Q}_0 &= \{Q_0'\} \\ \mathcal{Q}_{i+1} &= \mathcal{Q}_i \cup \{U \in \mathcal{P}(Q) \mid U = \Delta(S, a) \text{ for some } S \in \mathcal{Q}_i \text{ and } a \in A\}. \end{aligned}$$

the computation of $U = \Delta(S, a)$ can be done by computing first the set $W = \{q \in Q \mid (s, a, q) \in \theta \text{ for some } s \in S\}$ and then $U = K_{\mathcal{T}}(W)$. Stop when $\mathcal{Q}_{i+1} = \mathcal{Q}_i$. The set $\mathcal{Q}_i$ is the set of accessible states of $\mathcal{M}$. Output $\mathcal{M}' = \mathrm{ACC}(\mathcal{M})$, the accessible component of $\mathcal{M}$.

For the transition system



the transition system is: