Discrete Mathematics
Homework 1 Solutions

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Matt Lehman

September 15, 2014

Exercises

When you study these answers, look at the TEX too. There are some nice idioms here to learn.

1. How do the writing hints above apply particularly to you? (The point of this question is to force you to *read* those hints, rather than just skip them and start on the mathematics.)

   No common answer here. I may paste in some student responses.

2. Two interesting alphabets.

   (a) How many 10 letter words can be written with a 2 letter alphabet?

   We derived a formula for this problem, so all you need to do is apply it:

   With a 2 letter alphabet you can write $2^{10} = 1024$ words 10 letters long.

   Note that this answer is a complete English sentence, not just a number.

   You can also easily think through the answer from first principles. The alphabet might be $\{A, B\}$ or $\{0, 1\}$ or any other two element set. The key idea is that there are 2 choices for each of the 10 letters in the word, for a total of $2^{10} = 1024$. You do need to remember our convention that "words" may have repeated "letters".

   (b) How many 3 letter words can be written with a 10 letter alphabet?

   With a 10 letter alphabet you can write $10^3 = 1000$ words 3 letters long.

3. A number written in base 2 has about 3 times as many binary digits (bits) as the number of (decimal) digits in its base 10 representation.

   (a) Check this for a few interesting numbers.

   - Here are three small examples.
     The base 2 representation of 63 is 111111 (6 digits), 3 times as many binary digits.
     The base 2 representation of 64 is 1000000 (7 digits), 3.5 times as many binary digits.
     The base 2 representation of 127 is 1111111 (7 digits), about 2.33 times as many binary digits.
   - $1024_{10} = 10000000000_2$.[1] That's a four digit number in base 10 that takes 11 digits in base 2. If you think a little bit more about this example you'll note that 1024_{10} is just a little bit larger than the last three digit number, 999_{10}, while 2^{10} is just 1 more than the last binary ten digit number. So "three times as many digits" is a very good approximation here.

   (b) Explain why this follows from the answer to the previous question.

   We saw there that 2^{10} is relatively close to 10^3 – the error is just 0.24%. ("Kilo" in the metric system is 1,000 while "kilo" in computer science is 1,024, and the difference hardly ever matters.)

   A big number in base 10 is usually written this way

   `XXX,XXX,XXX,..., XXX,XXX`

[1] Subscripts are a standard convention indicate the base. The subscripts are in base 10.

Each three digit piece is an integer between 000 and 999, so can be thought of as a number in base 1000. Each of those three digit pieces will take about 10 digits in base 2, so if we're just looking for a rough approximation to convert from base 10 to base 2, we multiply the number of digits by $10/3 \approx 3$. Going in the other direction, we multiply the number of digits by $3/10 \approx 1/3$.

(c) Explain why this follows from the value of $\log_2(10)$ or $\log_{10}(2)$. (What is the relation between those two numbers?)

$\log_{10}(2) \approx 0.30103$. (That's a nice palindrome, easy to remember.) $\log_2(10) \approx 3.32 \approx 3\frac{1}{3}$, also easy to remember.

Each of those numbers is the reciprocal of the other – a fact you should know (from now on). How would you prove it?

In this problem what's important is the fact that

> The number of base b digits of n is (approximately) $\log_b(n)$.

Then

$$\log_2(n) = \log_2(10) \times \log_{10}(n) \approx 3.32 \log_{10}(n)$$

finishes the explanation.

4. Why is the number of $2n$ letter words on a k letter alphabet the square of the number of n letter words? You should be able to answer this two ways, one using a known formula, the other which works even if you have *no idea* what the formula is for n letter words.

First answer, using the formula.

$$\begin{aligned} \text{number of } 2n \text{ letter words} &= k^{2n} \\ &= (k^n)^2 \\ &= (\text{number of } n \text{ letter words})^2. \end{aligned}$$

Second answer. If you don't know the formula for the number of n letter words on a k letter alphabet (or know it but don't want to use it), then let that number, whatever it is, be W. A $2n$ letter word is one n letter word followed by another n letter word, so the answer would be $W \times W = W^2$.

5. Consider the word `APFELS`. [2]

- How many ways are there to rearrange these 6 letters?

  Each of the 6 letters is different, so this is the same as the number of 6 letter words on a 6 letter alphabet without repetitions, or the number of permutations of 6 things, so the answer is $6! = 720$.

- How many ways are there to rearrange these 6 letters so that a `P` comes before an `E`?

  Here are three ways to answer this question. I'll save the cleverest for last.

  – There are $(6 \times 5)/2 = 15$ ways to choose the two places for the `P` and the `E`. For each of those choices there are $4! = 24$ ways to arrange the other four letters. That means there are $15 \times 24 = 360$ ways to arrange all 6 as required.
  – If the `P` is first there are 5 places to put the `E`.
    If the `P` is second (third, fourth, fifth) there are 4 (3, 2, 1) places for the `E`.
    In each case there are and $4! = 24$ ways to fill in the other four letters, for a total of $(5 + 4 + 3 + 2 + 1) \times 24 = 360$ words.
  – There are $6! = 720$ words altogether. In just half of them the `P` precedes the `E`. That's 360 words.

6. Answer the questions in the last Exercise for the word `APPLES`.

There are $6!/2 = 360$ permutations of the letters. There are three equally likely patterns for the two `P`'s and the `E`, independent of how the other letters are arranged: `P.P.E`, `P.E.P` and `E.P.P`. A `P` precedes an `E` in two of them. That means 2/3 of the 360, so 240 words meet the requirements.

[2] That's "apples" in German.

7. How many permutations are there of the numbers $\{0, 1, \ldots, 99\}$? In how many of these do the numbers $\{1, 11, 21, \ldots, 91\}$ appear in order? Use Stirling's formula to estimate each of these counts.

   We're counting the number of ways of permuting 100 distinct objects. There are 100!, a very large number!

   In each of those, let's look at how the subset of $\{1, 11, 21, \ldots, 91\}$ might be arranged. There are 10 elements in that subset, so there are 10! ways of permuting those elements *with respect to each other*, only one of which gives us the desired result. Recall the second part of problem #5. We have to divide by the number of (equally likely) ways that the subset of 10 elements can be permuted, so the answer is 100!/10!.

   For $n = 100$ Stirling's formula tells us

$$100! \approx \sqrt{2\pi(100)} \left(\frac{100}{e}\right)^{100} \approx 9.325 * 10^{157}.$$

   Plugging 100! directly into a calculator yields

$$100! \approx 9.333 * 10^{157}$$

   so Stirling's is a pretty good approximation.

   10! isn't that large, so we don't need Stirling's formula to find it:

$$10! = 3,628,800 = 3.628 * 10^6$$

   so

$$\frac{100!}{10!} \approx \frac{9.333 * 10^{157}}{3.628 * 10^6} \approx 2.572 * 10^{151}.$$

8. Counting valid phone numbers.

   (a) In the 1960s a valid (North American) ten digit telephone number had the form `NYX NNX XXXX` where

      - `X` is one of the digits $0, 1, \ldots 9$,
      - `N` is one of the digits $2, 3, \ldots 9$,
      - `Y` is one of the digits $0, 1$.

      How many valid telephone numbers were there then?

      There were

$$\begin{aligned}(8)(2)(10)(8)(8)(10)(10)(10)(10)(10) &= (2)(8^3)(10^6)\\ &= (2^{10})(10^6)\\ &= 1,024,000,000\end{aligned}$$

      telephone numbers, or a little over 1 billion. I hope you didn't need a calculator to do that arithmetic.

   (b) Now the form is `NXX NXX XXXX`. How many valid numbers are there now?

      By direct count there are $(8^2)(10^8) = 6,400,000,000$, or 6.4 billion.

   (c) If you answered the previous question by a direct count, answer it again by thinking about how much larger it should be than the answer to the first question. If you did it that way first, answer it again with a direct count.

      If we compare the two strings: `NYX NNX XXXX` with `NXX NXX XXXX`, we only care about the digits whose type has changed.

      In the second place the change from `Y` to `X` increases the count by a factor of 10/2. In the fifth place the change from `N` to `X` increases the count by a factor of 10/8. The overall increase is by a factor of $(10/8) \times (10/2) = 100/16 = 6.25$. [3]

[3] Note that since $\frac{100}{16} = 6.25$, $\frac{1}{16} = 0.0625 = 6.25\%$ which is the current Massachusetts sales tax. If you buy something for $16 (more likely $15.95 or $15.99), the sales tax will be exactly $1. Massachusetts sales tax used to be 5%, which was much easier to calculate in your head.

(d) Were there enough numbers then? Are there enough numbers? Will there always be enough numbers? Write a paragraph or two about these questions. Use a round estimate for the population of North America.

In the 1960s, there were about 300 million people living in North America. If each household had a phone and there were an average of 4 people living in each household, only about 75 million numbers were needed for households. We also have to take into account businesses and pay phones, so if we want to be generous, we can probably round that figure to 200 million, so there were more than enough phone numbers.

Today, the population of North America is a little over 500 million. If every person has 2 phone numbers (cell phone and landline), this gives us a billion numbers. If we want to add in business phone numbers (I think we can forget about pay phones) then we can generously give everyone an average of 3 phone numbers, which is only 1.5 billion, just 25% of the total available. It seems unlikely that the population of North America will exceed a billion, and it also seems unlikely that everyone would need an average of more than 4 phone numbers, so it seems that this should be sufficient.

Here is the LaTeX source for this document. You can cut it from the pdf and use it to start your answers. I used the `\jobname` macro for the source file name, so you can call your file by any name you like.

```
% Math 320 hw1 solution
%
\documentclass{article}
\pagestyle{empty}
\usepackage[textheight=10in]{geometry}

\usepackage{amsmath}
\usepackage{hyperref}
\usepackage{graphicx}
\usepackage{verbatim}

\newcommand{\coursehome}
{http://www.cs.umb.edu/~eb/320}

\title{Discrete Mathematics \\
Homework 1 Solutions
}
\author{Ethan Bolker \\ Matt Lehman}

\begin{document}

\maketitle

\section*{Exercises}

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  too. There are some nice idioms here to learn.}

\begin{enumerate}

\item How do the writing hints above apply particularly to
  you? (The point of this question is to force you to \emph{read}
  those hints, rather than just skip them and start on the
  mathematics.)

No common answer here. I may paste in some student responses.

\item Two interesting alphabets.

\begin{enumerate}

\item How many $10$ letter words can be written with a $2$ letter
  alphabet?

We derived a formula for this problem, so all you need to do is apply
it:

With a $2$ letter alphabet you can write  $2^{10} = 1024$ words
$10$ letters long.

\emph{Note that this answer is a complete English sentence, not just a
  number.}

You can also easily think through the answer from first
principles. The alphabet might be $\{A,B\}$ or $\{0,1\}$ or any other
two element set. The key idea is that there are $2$ choices for each of
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the $10$ letters in the word, for a total of $2^{10} = 1024$. You do
need to remember our convention that ``words'' may have repeated
``letters''.

\item How many $3$ letter words can be written with a $10$ letter alphabet?

With a $10$ letter alphabet you can write
 $10^{3} = 1000$ words $3$ letters long.

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\item A number written in base $2$ has about $3$ times as many binary
  digits (bits) as the number of (decimal) digits in its base $10$
  representation.

\begin{enumerate}

\item Check this for a few interesting numbers.

\begin{itemize}

\item Here are three small examples.

The base $2$ representation of $63$ is $111111$ (6 digits), $3$ times
as many binary digits.

The base $2$ representation of $64$ is $1000000$ (7 digits), $3.5$
times as many binary digits.

The base $2$ representation of $127$ is $1111111$ (7 digits), about
$2.33$ times as many binary digits.


\item $1024_{10} = 10000000000_{2}$.
\footnote{
Subscripts are a standard convention indicate the base. The subscripts
are in base $10$.
}
That's a four digit number in
base $10$ that takes $11$ digits in base $2$. If you think a little
bit more about this example you'll note that $1024_{10}$ is just a
little bit larger than the last three digit number, $999_{10}$, while
$2^{10}$ is just $1$ more than the last binary ten digit number. So
``three times as many digits'' is a very good approximation here.

\end{itemize}

\item Explain why this follows from the answer to the  previous
  question.

We saw there that $2^{10}$ is relatively close to $10^{3}$ -- the
error is just $0.24\%$. (``Kilo'' in the metric system is 1,000 while
``kilo'' in computer science is 1,024, and the difference hardly ever
matters.)

A big number in base $10$ is usually written this way
%
\begin{verbatim}
               XXX,XXX,XXX,..., XXX,XXX
\end{verbatim} .
```

```
%
Each three digit piece is an integer between $000$ and $999$, so can
be thought of as a number in base $1000$. Each of those three digit
pieces will take about $10$ digits in base $2$, so if
we're just looking for a rough approximation to convert from base $10$ to base
$2$, we multiply the number of digits by $10/3 \approx 3$.  Going
in the other direction, we multiply the number of digits by
$3/10 \approx 1/3$.

\item Explain why this follows from the value of $\log_2(10)$ or
  $\log_{10}(2)$. (What is the relation between those two numbers?)

$\log_{10}(2) \approx 0.30103$. (That's a nice
  palindrome, easy to
  remember.)
$\log_{2}(10) \approx 3.32 \approx 3\frac{1}{3}$, also easy to remember.

Each of those numbers is the reciprocal of the other -- a fact you
should know (from now on). How would you prove it?

In this problem what's important is the fact that
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\begin{quotation}
The number of base $b$ digits of $n$ is (approximately) $\log_b(n)$.
\end{quotation}
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Then
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\begin{equation*}
\log_{2}(n) = \log_{2}(10) \times \log_{10}(n) \approx 3.32 \log_{10}(n)
\end{equation*}
%
finishes the explanation.
\end{enumerate}

\item Why is the number of $2n$ letter words on a $k$ letter alphabet
  the square of the number of $n$ letter words? You should be able to
  answer this two ways, one using a known formula, the other which works
  even if you have \emph{no idea} what the formula is for $n$ letter words.

First answer, using the formula.
% this commented line prevents the start of a new paragraph
\begin{align*}
\text{number of } 2n \text{ letter words}& = k^{2n} \\
& = (k^n)^2 \\
& = (\text{number of } n \text{ letter words})^2 .
\end{align*}

Second answer.  If you don't know the formula for the number of $n$
letter words on a $k$ letter alphabet (or know it but don't want to
use it), then let that number, whatever it is, be $W$.  A $2n$ letter
word is one $n$ letter word followed by another $n$ letter word, so
the answer would be $W \times W = W^{2}$.

\item Consider the word \verb!APFELS!.
\footnote{That's ``apples'' in German.}

\begin{itemize}
\item How many ways are there to rearrange these $6$ letters?
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Each of the $6$ letters is different, so this is the same as the
number of $6$ letter words on a $6$ letter alphabet without
repetitions, or the number of permutations of $6$ things,
so the answer is $6!= 720.$

\item How many ways are there to rearrange these $6$ letters so that
  a \verb!P! comes before an \verb!E!?

Here are three ways to answer this question. I'll save the cleverest
for last.

\begin{itemize}

\item There are $(6 \times 5)/{2} = 15$ ways to choose the two places
  for the \verb!P! and the \verb!E!. For each of those choices there
  are $4! = 24$ ways to arrange the other four letters. That means
  there are $15 \times 24 = 360$ ways to arrange all $6$ as required.

\item If the \verb!P! is first there are $5$ places to put the
  \verb!E!.

If the \verb!P! is second (third, fourth, fifth)  there
  are $4$ ($3$, $2$, $1$) places for the \verb!E!.

In each case there are  and $4! = 24$ ways to fill in the other four
letters, for a total of $(5 + 4 + 3 + 2 + 1) \times 24 = 360$ words.

\item There are $6! = 720$ words altogether. In just half of them the
  \verb!P! precedes the \verb!E!. That's $360$ words.
\end{itemize}
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\item Answer the questions in the last Exercise for the word
  \verb!APPLES!.

There are $6!/2 = 360$ permutations of the letters.
There are three equally likely patterns for the two \verb!P!'s and the
\verb!E!, independent of how the other letters are arranged:
\verb!P.P.E!,
\verb!P.E.P! and
\verb!E.P.P!.
A \verb!P! precedes an \verb!E! in two of them.
That means $2/3$ of the $360$, so $240$ words meet the requirements.


\item How many permutations are there of the numbers $\{0, 1, \ldots,
  99\}$? In how many of these do the numbers $\{1, 11, 21, \ldots,
  91\}$ appear in order? Use Stirling's formula to estimate each of
  these counts.

% this is just C(100,10), derived a new way ...

We're counting the number of ways of permuting $100$ distinct
objects. There are $100!$, a very large number!

In each of those, let's look at how the subset of $\{1, 11, 21,
\ldots,91\}$ might be  arranged.  There are $10$
elements in that subset, so there are $10!$ ways of permuting those
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the desired result.  Recall the second part of problem \#5.  We have
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to divide by the number of (equally likely)
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\item Counting valid phone numbers.

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  number had the form \verb!NYX NNX XXXX! where

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\item \verb!X! is one of the digits $0, 1, \ldots 9$,
\item \verb!N! is one of the digits $2, 3, \ldots 9$,
\item \verb!Y! is one of the digits $0, 1$.
\end{itemize}

How many valid telephone numbers were there then?

There were
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\begin{align*}
(8)(2)(10)(8)(8)(10)(10)(10)(10)(10) & = (2)(8^{3})(10^{6}) \\
& = (2^{10})(10^{6}) \\
& =  1,024,000,000
\end{align*}
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 telephone numbers, or a little over $1$ billion. I hope you didn't
 need a calculator to do that arithmetic.

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America.  If each household had a phone and there were an average of
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\newpage

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\verbatiminput{\jobname}

\end{document}
```