Discrete Mathematics
Homework 2

Ethan Bolker

September 26, 2014

Due: Tuesday, September 30.

Preliminaries

- Here's how to write binomial coefficients in LaTeX:

  `{a \choose b}`: ${a \choose b}$ `\binom{a}{b}`: $\binom{a}{b}$

- Some of these exercises are routine. Some are almost baby research problems. Do the best you can. I don't expect anyone to do them all!
- Truth in advertising. Some of the routine problems come from several standard Discrete Mathematics texts. I haven't credited them individually since they don't really represent significant intellectual effort on the part of their authors.
- This quote From G. K. Chesterton's *The Scandal of Father Brown* may help you remember that the start of every counting problem is understanding exactly what it is you have been asked to count. Work out some small cases by brute force – don't just guess a formula and hope you're right.

  > Father Brown laid down his cigar and said carefully: "It isn't that they can't see the solution. It is that they can't see the problem." [1]

- Submit LaTeX source as usual as part of your document. Be sure to use hard carriage returns so I can read all your source code. I haven't done that here! [2]
- Delete these preliminaries.

Exercises

1. Study the posted solutions to the first homework. Then write a sentence or two (or a paragraph or two) explaining what you will do differently this time based on what you discovered there.
2. How many permutations are there of the six letters in the word `APPLES` ?

   We've seen this word before. To answer the question, first find out how many ways there are to place the two `P`s.
3. How many permutations are there of the letters in the word `BOOKKEEPER` ?
4. How many permutations are there of the letters in the word `BOOKKEEPER` in which the two `O`s and the two `K`s are adjacent (but not necessarily next to each other).
5. How many permutations are there of the string `ABCDEFGGG` in which exactly two of the `G`s are adjacent?
6. How many ways can b books be placed on s numbered shelves

[1] `http://gutenberg.net.au/ebooks02/0201031.txt`
[2] Why is "enter" on your keyboard sometimes called "carriage return"?

(a) if the books are indistinguishable copies of the same title?

(b) if no two books are the same, but the positions of the books on each shelf don't matter?

(c) if no two books are the same, and the positions of the books on each shelf matter?

7. How many ways are there to deal hands of seven cards to each of five players from a standard deck of 52 cards?

8. How many subsets of a 20 element set contain exactly 4 elements? How many contain at most 4 elements?

9. If you flip a fair coin 20 times, what is the probability that you get exactly 4 heads? At most 4 heads?

10. How many numbers less than one million have exactly four ones in their binary representation? What are the largest and smallest such numbers?

11. The roads in a city are laid out on a square grid with numbered Streets and Avenues, so each corner is defined as the intersection of a Street and an Avenue. For example, $(2, 4)$ is the corner of Second Street and Fourth Avenue.

    Let $T(m, n)$ be the number of ways to walk from $(1, 1)$ to (m, n) so that at each stage either the Street number or the Avenue number increases. For example, here are two of the $T(2, 4) = 4$ ways to get to $(2, 4)$:

$$(1,1) \to (1,2) \to (2,2) \to (2,3) \to (2,4)$$

$$(1,1) \to (1,2) \to (1,3) \to (2,3) \to (2,4)$$

    Find a closed form expression for $T(m, n)$. You may want to start by computing some small examples.

12. An identity. Let

$$M(n) = \sum_{j=1}^{n} j \binom{n}{j}.$$

    That identity is correct – because I fixed a typo. This was the original:

$$M(n) = \sum_{j=1}^{n} n \binom{n}{j}.$$

    - Calculate the first few values of M.
    - Guess the pattern. Here's a hint: look at $M(n+1)/2M(n)$.
    - Guess a closed form expression for $M(n)$.
    - Prove that your guess is correct. Here are hints for two different proofs - do both if you can. (1) Use the binomial theorem and differentiate $(1+x)^n$. (2) Try induction.

13. From the symmetry of the binomial coefficients, it is not too hard to see that when n is an odd number, the number of subsets of $\{1, 2, \ldots, n\}$ of odd size equals the number of subsets of even size. Is that true when n is even? Why or why not?

    Hint: Think about a recursive/inductive condition like the one we used in class to count the total number of subsets of an n element set when we knew the answer for an $n - 1$ element set.

14. Apples and bananas

    Kids in first grade spend a fair amount of time mastering addition of small numbers by answering questions like

    > If you have some apples and some bananas and you have 10 pieces of fruit altogether, how many of each might you have?

    They work at listing all the possible answers to this question. They use 10 more often than any other total since combinations making 10 are the most important when learning arithmetic.

(a) How many ways solutions are there for n pieces of fruit?

(b) The kids often ask (as they should!) whether it's OK to have *no* apples. Which answer makes the most mathematical sense?

(c) Answer the first question both when 0 is allowed as a summand and when it's not.

15. Apples, bananas and cherries.

    Ask and answer the questions in the previous exercise for three kinds of fruit.

    Hint: The hint at Problem 13 may help here too. Think about solutions that use no apples.

    Hint: Do you recognize any binomial coeffiecients? (The numbers in the first few rows of Pascal's triange should be your friends.)

16. Apples, bananas, cherries, dates, eggplants, ...

    - How many ways are there to write n as the sum of an ordered list of k nonnegative integers?
    - How many ways are there to write n as the sum of an ordered list of k positive integers?
    - (Trick question) How many ways are there to write n as the sum of an ordered list of k integers?

17. The previous questions are all *much harder* if the list isn't ordered. Then $3 + 7$ and $7 + 3$ count as the same way to sum to 10. Each way to write n is a *partition* of n.

    Let $P(n)$ be the number of partitions of n. Find the first few values of $P(n)$ and then look up the sequence at the On-Line Encyclopedia of Integer Sequences (`http://oeis.org/` and `http://oeis.org/wiki/Welcome`).

```latex
% Math 320 hw2
%
\documentclass{article}
\pagestyle{empty}
\usepackage[textheight=10in]{geometry}

\usepackage{amsmath}
\usepackage{hyperref}
\usepackage{graphicx}
\usepackage{verbatim}

\newcommand{\coursehome}
{http://www.cs.umb.edu/~eb/320}

\title{Discrete Mathematics \\
Homework 2
}
\author{Ethan Bolker}
%\date{September 1, 2014}

\begin{document}

\maketitle

\noindent
Due: Tuesday, September 30.

\section*{Preliminaries}

\begin{itemize}

\item Here's how to write binomial coefficients in \LaTeX:
%
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\verb! \binom{a}{b}!: $\binom{a}{b}$
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  understanding exactly what it is you have been asked to count. Work
  out some small cases by brute force -- don't just guess a formula
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\end{quotation}
```

```
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\section*{Exercises}

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Then write a sentence or two (or a paragraph or two) explaining what
you will do differently this time based on what you discovered there.

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\item The roads in a city are laid out on a square grid with numbered
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increases. For example, here are two of the $T(2,4) = 4$ ways to get
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% This commented line tells TeX not to start a new paragraph
\begin{equation*}
(1,1) \rightarrow
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%
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Find a closed form expression for $T(m,n)$. You may want to start by
computing some small examples.

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Let
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\emph{That identity is correct -- because I fixed a typo. This was the
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\begin{itemize}
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\item Guess a closed form expression for $M(n)$.
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  proofs - do both if you can. (1) Use the binomial theorem and differentiate
  $(1+x)^n$. (2) Try induction.
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\item \label{problem:oddandevensubsets}
 From the symmetry of the binomial coefficients, it is not too
  hard to see that when $n$ is an odd number, the number of subsets of
$\{1,2, \ldots, n\}$ of odd
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even size. Is that true when $n$ is even?
Why or why not?

\emph{Hint:} Think about a recursive/inductive condition like the one
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  and when it's not.
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\item Apples, bananas and cherries.

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of fruit.


\emph{Hint:} The hint at Problem~\ref{problem:oddandevensubsets} may
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in the first few rows of Pascal's triange should be your friends.)

\item Apples, bananas, cherries, dates, eggplants, \ldots

\begin{itemize}

\item How many ways are there to write $n$ as the sum of an ordered list of
$k$ nonnegative integers?

\item How many ways are there to write $n$ as the sum of an ordered list of
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\item (Trick question) How many ways are there to write $n$ as the sum
  of an ordered list of $k$ integers?
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  isn't ordered. Then $3+7$ and $7+3$ count as the same way to sum to
  $10$.  Each way to write $n$ is a \emph{partition} of $n$.

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\end{enumerate}
```

```
\newpage

\verbatiminput{\jobname}

\end{document}
```