Discrete Mathematics
Homework 4

Ethan Bolker

November 30, 2014

Due: ???

This homework covers problems in an area loosely called "sets and functions". You can read about it in B&W at `http://cseweb.ucsd.edu/~gill/BWLectSite/Resources/C1U4SF.pdf`.

I will add to this homework from time to time.

See the source code for some cool `\set` macros to construct these sets in LaTeX:

$$D = \{x \in \mathbb{N} \,|\, 1 \le x \le 100\}$$

The delimiters adjust to the size of the contents in the * version:

$$E = \left\{ x \in \mathbb{Q} \,\middle|\, -\frac{1}{2} \le x \le \frac{1}{2} \right\}$$

You also can have a manual adjustment with an optional argument to `\set`:

$$E = \big\{ x \in \mathbb{Q} \,\big|\, -\tfrac{1}{2} \le x \le \tfrac{1}{2} \big\}$$

And you can define sets as simple lists:

$$\text{Unit fractions} = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots \right\}$$

Exercises

1. Is forming the cartesion product of two sets a commutative operation?

   Let S and T be sets. Investigate when/whether

   $$S \times T = T \times S.$$

   Start this problem by working a few small examples! When you understand what's going on, write your answer as a theorem like this:

   Theorem 1. *$S \times T = T \times S$ if and only if [something] or [something else].*

   Proof. Suppose $S \times T = T \times S$. Then ... so something or something else is true.

   Conversely, suppose neither something nor something else is true. Then ... so $S \times T \ne T \times S$. □

2. The algebra of symmetric differences.

   We'll use $A \oplus B$ for the symmetric difference of subsets A and B of a universe U.

   - Show that $\oplus$ is associative.
   - Find a subset $I \subseteq U$ such that for every A we have $A \oplus I = A$.
   - For each A, find a B such that $A \oplus B = I$.
   - True or false:

     If $A \oplus B = A \oplus C$ then $B = C$.

     Just saying "true" (or "false") isn't an answer. You must explain why.

Hints:

- If you have taken or are taking abstract algebra lots of this should look familiar.
- Thinking about how $\oplus$ works with the bit vector representation of subsets might save you a lot of thinking and writing.

3. A binary relation on a set may be reflexive, symmetric or transitive. Call those properties R, S, and T for short. There are eight possible truth value combinations for those properties. For each of the eight, find an example. (So, for example, one of your answers should exhibit a binary relation that satisfies R and $\sim S$ and $\sim T$.)

   Please find elegant examples – the best are either everyday situations where the properties are obviously what you claim they are, or binary relations built on the smallest possible finite set that will do the job. You can use answers to previous problems when they work here.

4. Let $\star$ be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by

$$(a, b) \star (c, d) \iff ad = bc.$$

   (a) Describe all the pairs (a, b) such that $(a, b) \star (4, 6)$.
   (b) Show that $\star$ is not an equivalence relation, but is an equivalence relation if you consider only the ordered pairs in which the second coordinate is not 0.
   (c) Prove: given (a, b) with $b \neq 0$ there is a unique pair (m, n) equivalent to (a, b) for which $\gcd(m, n) = 1$. (Hint, if needed: think about the previous part of the problem.)
   (d) What happens to the previous claim when $b = 0$?
   (e) Show that there is a bijection betwee $\mathbb{Q}$ (the rational numbers) and the partition corresponding to $\star$ when the second coordinate is not zero.

5. For each of the following sets, provide an argument showing that it's countable, or that it's not.

   (a) The set of all partitions of $\mathbb{N}$.
   (b) The set of all partitions of $\mathbb{N}$ into a finite number of blocks.
   (c) The set of all partitions of $\mathbb{N}$ for which each block is finite. [1]

6. Counting computer programs

   Each of these questions can be answered in a sentence or two if you really understand the work we did in class on counting infinities.

   (a) Show that there are only countably many finite strings of (ascii) characters.
   If you can't show this, say so. Then assume it's true and do the rest of the problem.
   (b) Show that in any particular programming language there are only countably many computer programs that accept an integer as input and produce a boolean value as output.
   (c) Show that there are uncountably many functions from $\mathbb{Z}$ to $\{T, F\}$.
   (d) Show that for any particular computer language there are functions from $\mathbb{Z}$ to $\{T, F\}$ that can't be implemented by a program in that language.
   (e) If you have a language and a function that can't be computed using that language you can create a new and better language to compute it. If you do that over and over again will you have a language that computes all functions?

7. (Optional, but I'd love to see it.)

   Read the fictional treatment of Hilbert's Hotel at `http://www.c3.lanl.gov/mega-math/workbk/infinity/inhotel.html`.

   Then rewrite the ending so that the narrator's scheme for housing the people on the infinitely many infinite busses succeeds (as it will). Then have the fire caused in some interesting way by the failure (discussed in class) of any scheme that claims to accomodate all the committees of guests.

   Try to copy the author's style.

[1] I just made up this example. I don't know the answer yet.

```
% Math 320 hw4
%
\documentclass{article}
\pagestyle{empty}
\usepackage[textheight=10in]{geometry}

\usepackage{mathtools, nccmath}

\usepackage{amssymb}
\usepackage{amsthm}
\usepackage{listings}
\usepackage{graphicx}
\usepackage{verbatim}

% hyperref should be (nearly) the last package loaded
\usepackage{hyperref}

\usepackage{xparse}

%This very cool macro comes from
%http://tex.stackexchange.com/questions/209863/how-to-add-mathematical-notation-of-a-set
%
% \set{ stuff ; something }
% expands to
% { stuff | something }
%
\DeclarePairedDelimiterX{\set}[1]{\{}{\}}{\setargs{#1}}
\NewDocumentCommand{\setargs}{>{\SplitArgument{1}{;}}m}
{\setargsaux#1}
\NewDocumentCommand{\setargsaux}{mm}
{\IfNoValueTF{#2}{#1} {#1\,\delimsize|\,\mathopen{}#2}}%{#1\:;\:#2}
\newtheorem{theorem}{Theorem}

\newcommand{\coursehome}
{http://www.cs.umb.edu/~eb/320}

\title{Discrete Mathematics \\
Homework 4
}
\author{Ethan Bolker}
%\date{September 1, 2014}

\newcommand{\ZZ}{\mathbb{Z}}
\newcommand{\NN}{\mathbb{N}}
\newcommand{\QQ}{\mathbb{Q}}


%create (mod n) macro
\newcommand{\mm}[1]{%
\ensuremath{(\text{mod } #1)}}

\begin{document}


\maketitle

\noindent
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See the source code for some cool \verb+\set+ macros to construct
these sets in \LaTeX:

\[
D = \set{x \in \NN ; 1\leq x\leq 100}
\]

The delimiters adjust to the size of the contents in the * version:
\[
E = \set*{x \in \QQ ; -\frac{1}{2}\leq x \leq \frac{1}{2}}
\]

You also can have a manual adjustment with an optional argument to \verb+\set+:
\[
E = \set[\big]{x \in \QQ ; -\mfrac{1}{2}\leq x \leq \mfrac{1}{2}}
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And you can define sets as simple lists:
\[\text{Unit fractions}= \set*{\mfrac{1}{1}, \mfrac{1}{2},\mfrac{1}{3}, \dots }
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\section*{Exercises}

\begin{enumerate}

\item Is forming the cartesion product of two sets a commutative
  operation?

Let $S$ and $T$ be sets. Investigate when/whether
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\begin{equation*}
S \times T = T \times S .
\end{equation*}

Start this problem by working a few small examples! When you
understand what's going on, write your answer as a theorem like this:

\begin{theorem}
$S \times T = T \times S$  if and only if [something] or [something else].
\end{theorem}
\begin{proof}
Suppose $S \times T = T \times S$. Then \ldots so something or
something else is true.

Conversely, suppose neither something nor something else is true. Then
\ldots so
$S \times T \not = T \times S$.
\end{proof}

\item The algebra of symmetric differences.

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$B$ of a universe $U$.
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\begin{itemize}
\item Show that $\oplus$ is associative.
\item Find a subset $I \subseteq U$ such that for every $A$ we have $A
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\item True or false:
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\begin{equation*}
\text{If } A \oplus B = A \oplus C \text{ then } B = C.
\end{equation*}

Just saying ‘‘true’’ (or ‘‘false’’) isn’t an answer. You must explain
why.
\end{itemize}

Hints:
\begin{itemize}
\item If you have taken or are taking abstract algebra lots of this
  should look familiar.

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  representation of subsets might save you a lot of thinking and
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\end{enumerate}
\newpage

\verbatiminput{\jobname}
```

```
\end{document}
```