

Discrete Mathematics

Homework 5

Ethan Bolker

December 1, 2014

Due: ???

This homework covers problems about graphs, and some about games. You can read about graphs in B&W at <http://cseweb.ucsd.edu/~gill/BWLectSite/Resources/C2U4GT.pdf>.

You can read about games at <http://www.cs.umb.edu/~eb/320/FergusonGames.pdf>.

I will add to this homework from time to time.

If you want to include figures – for example, sketches of graphs – you can draw them by hand and scan them, or learn tikz, or if necessary, turn in hand drawn sketches.

Exercises

1. Prove that in any finite graph the number of vertices of odd degree is even.
2. *Dirac's Theorem* says that if in a *simple* graph with $n \geq 3$ every vertex has degree at least $n/2$ then the graph has a Hamiltonian cycle.

That Dirac is *not* the Nobel Prize winning physicist P. A. M. Dirac.

A *simple* graph is an undirected graph with no loops and no multiple edges.

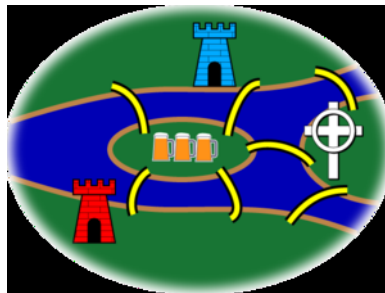
Show that the converse of Dirac's Theorem is false for graphs with at least 3 vertices.

3. Königsberg updated

Here is a story that starts with the classic problem and asks some questions. Answer them.

I didn't make this up. It's very easy to find on the internet, along with a solution. But the questions are easy, so please don't look up the answers.

Here's the picture:



The northern bank of the river is occupied by the Schloss, or castle, of the Blue Prince; the southern by that of the Red Prince. The east bank is home to the Bishop's Kirche, or church; and on the small island in the center is a Gasthaus, or inn.

It is understood that the problems to follow should be taken in order, and begin with a statement of the original problem:

It being customary among the townsmen, after some hours in the Gasthaus, to attempt to walk the bridges, many have returned for more refreshment claiming success. However, none have been able to repeat the feat by the light of day.

The Blue Prince, having analyzed the town's bridge system by means of graph theory, concludes that the bridges cannot be walked. He contrives a stealthy plan to build an eighth

bridge so that he can begin in the evening at his Schloss, walk the bridges, and end at the Gasthaus to brag of his victory. Of course, he wants the Red Prince to be unable to duplicate the feat from the Red Castle. Where does the Blue Prince build the eighth bridge?

The Red Prince, infuriated by his brother's Gordian solution to the problem, wants to build a ninth bridge, enabling him to begin at his Schloss, walk the bridges, and end at the Gasthaus to rub dirt in his brother's face. As an extra bit of revenge, his brother should then no longer be able to walk the bridges starting at his Schloss and ending at the Gasthaus as before. Where does the Red Prince build the ninth bridge?

The Bishop has watched this furious bridge-building with dismay. It upsets the town's Weltanschauung and, worse, contributes to excessive drunkenness. He wants to build a tenth bridge that allows all the inhabitants to walk the bridges and return to their own beds. Where does the Bishop build the tenth bridge?

Two more questions:

- Most of the German words in this passage are translated there. One is not. What does "Weltanschauung" mean? Please explain in your own words. Don't just paste in a dictionary definition.
- Why does the Red Prince call his brother's solution "Gordian"?

4. Generalizing Euler's Theorem

Euler's theorem for planar connected graphs says

$$F - E + V = 1$$

where F is the number of regions, E the number of edges and V the number of vertices.

Discover and then prove a generalization for a planar graph with C connected components. Your formula should reduce to Euler's when $C = 1$.

5. Nonplanar graphs

On page 124 of David Richeson's *Euler's Gem* (<http://www.eulersgem.com/>) you can find the following proof that K_5 , the complete graph on five points, is not planar.

Suppose K_5 is a planar graph. Then we can draw K_5 in the plane so that no edges cross. K_5 has 5 vertices and 10 edges. Euler's formula for planar graphs states that $V - E + F = 2$ [counting the outside of the graph as one of the faces], thus our planar drawing of K_5 must have 7 faces including the unbounded face (because $2 = 5 - 10 + F$).

Each edge borders two faces, so $2E = pF$, where p is the average number of sides on all faces. K_5 is a complete graph, so it has no loops or parallel edges. Because there are no loops, there are no faces bounded by only one edge, and because there are no parallel edges there are no faces bounded by two edges. Thus the average number of edges per face is at least three. So, $p \geq 3$ and $2E \geq 3F$. But $F = 7$ and $E = 10$ implies that $20 \geq 21$, which is a contradiction. It must be that K_5 is non planar.

In a similar way we can prove that the complete bipartite graph $K_{3,3}$ is not planar (give it a try!). The key difference is that because $K_{3,3}$ is bipartite, a path that begins and ends at the same vertex must have an even number of edges. So there can be no three-sided faces either.

Your job: complete the proof suggested in the last paragraph.

```

% Math 320 hw5
%
\documentclass{article}
\pagestyle{empty}
\usepackage[textheight=10in]{geometry}

\usepackage{mathtools, nccmath}

\usepackage{amssymb}
\usepackage{amsthm}
\usepackage{listings}
\usepackage{graphicx}
\usepackage{verbatim}

% hyperref should be (nearly) the last package loaded
\usepackage{hyperref}

\usepackage{xparse}

%This very cool macro comes from
%http://tex.stackexchange.com/questions/209863/how-to-add-mathematical-notation-of-a-set
%
% \set{ stuff ; something }
% expands to
% { stuff | something }
%
\DeclarePairedDelimiterX{\set}[1]{\{ }\}{\setargs{#1}}
\NewDocumentCommand{\setargs}{>\SplitArgument{1}{;}m}
{\setargsaux#1}
\NewDocumentCommand{\setargsaux}{mm}
{\IfNoValueTF{#2}{#1} {#1\,\delimsize|\,\mathopen{ }#2}}%{#1\;\;:\;\;#2}
\newtheorem{theorem}{Theorem}

\newcommand{\coursehome}
{http://www.cs.umb.edu/~eb/320}

\title{Discrete Mathematics \\  
Homework 5
}
\author{Ethan Bolker}
%\date{September 1, 2014}

\newcommand{\ZZ}{\mathbb{Z}}
\newcommand{\NN}{\mathbb{N}}
\newcommand{\QQ}{\mathbb{Q}}

%create (mod n) macro
\newcommand{\mm}[1]{%
\ensuremath{(\text{mod } #1)}}

\begin{document}

\maketitle

\noindent
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`\section*{Exercises}`

`\begin{enumerate}`

`\item` Prove that in any finite graph the number of vertices of odd degree is even.

`\item` `\emph{Dirac's Theorem}` says that if in a `\emph{simple}` graph with $n \geq 3$ every vertex has degree at least $n/2$ then the graph has a Hamiltonian cycle.

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```
\begin{center}
\includegraphics[width= 2in]{konigsbergplus}
\end{center}
```

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\begin{itemize}
```

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```

```
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```
\item Generalizing Euler's Theorem
```

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```
\begin{equation*}
```

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%

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```
\begin{quotation}
```

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`\end{enumerate}`

`\newpage`

`\verbatiminput{\jobname}`

`\end{document}`