Discrete Mathematics
Homework 5

Ethan Bolker

December 11, 2014

Due: Last class, December 11.

This homework covers problems about graphs, and some about games. You can read about graphs in B&W at `http://cseweb.ucsd.edu/~gill/BWLectSite/Resources/C2U4GT.pdf`.

You can read about games at `http://www.cs.umb.edu/~eb/320/FergusonGames.pdf`.

If you want to include figures – for example, sketches of graphs – you can draw them by hand and scan them, or learn tikz, or if necessary, turn in hand drawn sketches.

Exercises

1. Prove that in any finite graph the number of vertices of odd degree is even.

   Solution

   When I add up the degrees of the vertices each edge will be counted twice, so the sum will be even. If a sum of numbers is even then there must be an even number of odds.

2. *Dirac's Theorem* says that if in a *simple* graph with $n \geq 3$ every vertex has degree at least $n/2$ then the graph has a Hamiltonian cycle.

   That Dirac is *not* the Nobel Prize winning physicist P. A. M. Dirac.

   A *simple* graph is an undirected graph with no loops and no multiple edges.

   Show that the converse of Dirac's Theorem is false for graphs with at least 3 vertices.

   Solution

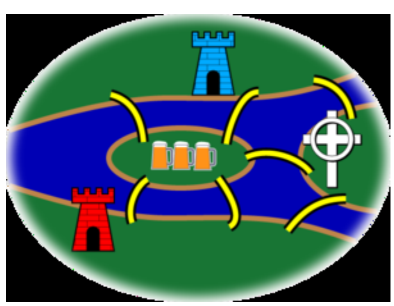
   This is a little hard to parse. The converse is false because the existence of a Hamiltonian cycle doesn't force $n/2$ vertices. Any *cycle* of length five or more provides a counterexample.

3. Königsberg updated

   Here is a story that starts with the classic problem and asks some questions. Answer them.

   I didn't make this up. It's very easy to find on the internet, along with a solution. But the questions are easy, so please don't look up the answers.

   Here's the picture:

   The northern bank of the river is occupied by the Schloss, or castle, of the Blue Prince; the southern by that of the Red Prince. The east bank is home to the Bishop's Kirche, or church; and on the small island in the center is a Gasthaus, or inn.

   It is understood that the problems to follow should be taken in order, and begin with a statement of the original problem:

It being customary among the townsmen, after some hours in the Gasthaus, to attempt to walk the bridges, many have returned for more refreshment claiming success. However, none have been able to repeat the feat by the light of day.

The Blue Prince, having analyzed the town's bridge system by means of graph theory, concludes that the bridges cannot be walked. He contrives a stealthy plan to build an eighth bridge so that he can begin in the evening at his Schloss, walk the bridges, and end at the Gasthaus to brag of his victory. Of course, he wants the Red Prince to be unable to duplicate the feat from the Red Castle. Where does the Blue Prince build the eighth bridge?

The Red Prince, infuriated by his brother's Gordian solution to the problem, wants to build a ninth bridge, enabling him to begin at his Schloss, walk the bridges, and end at the Gasthaus to rub dirt in his brother's face. As an extra bit of revenge, his brother should then no longer be able to walk the bridges starting at his Schloss and ending at the Gasthaus as before. Where does the Red Prince build the ninth bridge?

The Bishop has watched this furious bridge-building with dismay. It upsets the town's Weltanschauung and, worse, contributes to excessive drunkenness. He wants to build a tenth bridge that allows all the inhabitants to walk the bridges and return to their own beds. Where does the Bishop build the tenth bridge?

Two more questions:

- Most of the German words in this passage are translated there. One is not. What does "Weltanschauung" mean? Please explain in your own words. Don't just paste in a dictionary definition.
- Why does the Red Prince call his brother's solution " Gordian"?

Solution

For the puzzle, see the Variations on the Wikipedia page `http://en.wikipedia.org/wiki/Seven_Bridges_of_K%C3%B6nigsberg`.

The web will answer the other two questions too.

4. Generalizing Euler's Theorem

Euler's theorem for planar connected graphs says

$$F - E + V = 1$$

where F is the number of regions, E the number of edges and V the number of vertices.

Discover and then prove a generalization for a planar graph with C connected components. Your formula should reduce to Euler's when $C = 1$.

Solution

For a planar graph

$$F - E + V - C = 0$$

where C is the number of components. This is pretty easy to prove. If you use Euler's formula for each component and add the resulting equations you get C on the right hand side. I just moved it to the left as $-C$ for extra elegance.

5. Nonplanar graphs

On page 124 of David Richeson's *Euler's Gem* (`http://www.eulersgem.com/` you can find the following proof that K_5, the complete graph on five points, is not planar.

> Suppose K_5 is a planar graph. Then we can draw K_5 in the plane so that no edges cross. K_5 has 5 vertices and 10 edges. Euler's formula for planar graphs states that $V - E + F = 2$ [counting the outside of the graph as one of the faces], thus our planar drawing of K_5 must have 7 faces including the unbounded face (because $2 = 5 - 10 + F$).
>
> Each edge borders two faces, so $2E = pF$, where p is the average number of sides on all faces. K_5 is a complete graph, so it has no loops or parallel edges. Because there are no loops, there are no faces bounded by only one edge, and because there are no parallel edges there are no faces bounded by two edges. Thus the average

number of edges per face is at least three. So, $p \geq 3$ and $2E \geq 3F$. But $F = 7$ and $E = 10$ implies that $20 \geq 21$, which is a contradiction. It must be that K_5 is non planar.

In a similar way we can prove that the complete bipartite graph $K_{3,3}$ is not planar (give it a try!). The key difference is that because $K_{3,3}$ is bipartite, a path that begins and ends at the same vertex must have an even number of edges. So there can be no three-sided faces either.

Your job: complete the proof suggested in the last paragraph.

Solution

Suppose $K_{3,3}$ is a planar graph. Then we can draw $K_{3,3}$ in the plane so that no edges cross. $K_{3,3}$ has 6 vertices and 9 edges. Euler's formula for planar graphs states that $V - E + F = 2$ [counting the outside of the graph as one of the faces], thus our planar drawing of $K_{3,3}$ must have 5 faces including the unbounded face (because $2 = 6 - 9 + F$).

Each edge borders two faces, so $2E = pF$, where p is the average number of sides on all faces. $K_{3,3}$ has no loops or parallel edges. Because there are no loops, there are no faces bounded by only one edge, and because there are no parallel edges there are no faces bounded by two edges. Because it is bipartite, every cycle has even length, so there are no faces bounded by three edges. Thus the average number of edges per face is at least four. So, $p \geq 4$ and $2E \geq 4F$. But $F = 5$ and $E = 9$ implies that $18 \geq 20$, which is a contradiction. It must be that $K_{3,3}$ is non planar.

```latex
% Math 320 hw5
%
\documentclass{article}
\pagestyle{empty}
\usepackage[textheight=10in]{geometry}

\usepackage{mathtools, nccmath}

\usepackage{amssymb}
\usepackage{amsthm}
\usepackage{listings}
\usepackage{graphicx}
\usepackage{verbatim}

% hyperref should be (nearly) the last package loaded
\usepackage{hyperref}

\usepackage{xparse}

%This very cool macro comes from
%http://tex.stackexchange.com/questions/209863/how-to-add-mathematical-notation-of-a-set
%
% \set{ stuff ; something }
% expands to
% { stuff | something }
%
\DeclarePairedDelimiterX{\set}[1]{\{}{\}}{\setargs{#1}}
\NewDocumentCommand{\setargs}{>{\SplitArgument{1}{;}}m}
{\setargsaux#1}
\NewDocumentCommand{\setargsaux}{mm}
{\IfNoValueTF{#2}{#1} {#1\,\delimsize|\,\mathopen{}#2}}%{#1\:;\:#2}
\newtheorem{theorem}{Theorem}

\newcommand{\coursehome}
{http://www.cs.umb.edu/~eb/320}

\title{Discrete Mathematics \\
Homework 5
}
\author{Ethan Bolker}
%\date{September 1, 2014}

\newcommand{\ZZ}{\mathbb{Z}}
\newcommand{\NN}{\mathbb{N}}
\newcommand{\QQ}{\mathbb{Q}}


%create (mod n) macro
\newcommand{\mm}[1]{%
\ensuremath{(\text{mod } #1)}}

\begin{document}

\maketitle

\noindent
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\section*{Exercises}

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\item Prove that in any finite graph the number of vertices of odd
  degree is even.

\textbf{Solution}

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will be counted twice, so the sum will be even. If a sum of numbers is
even then there must be an even number of odds.

\item \emph{Dirac's Theorem} says that if in a \emph{simple} graph
  with $n \ge 3$ every vertex has degree at least $n/2$ then the graph
  has a Hamiltonian cycle.

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\begin{center}
\includegraphics[width= 2in]{konigsbergplus}
\end{center}

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Please explain in your own words. Don’t just paste in a dictionary
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\item Why does the Red Prince call his brother’s solution ‘‘
  Gordian’’?

\end{itemize}

\textbf{Solution}

For the puzzle, see the Variations on the Wikipedia page
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where $F$ is the number of regions, $E$ the number of edges and $V$ the
number of vertices.
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connected components. Your formula should reduce to Euler's when $C=1$.

\textbf{Solution}

For a planar graph
\begin{equation*}
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\end{equation*}
where $C$ is the number of components. This is pretty easy to
prove. If you use Euler's formula for each component and add the
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On page 124 of David Richeson's
\emph{Euler's Gem} (\url{http://www.eulersgem.com/} you can find the
following proof that $K_5$, the complete graph on five points, is not
planar.


\begin{quotation}
Suppose $K_5$ is a planar graph. Then we can draw
$K_5$ in the plane so that no edges cross. $K_5$ has $5$
vertices and $10$ edges. Euler's formula for planar graphs states that
$V-E+F=2$ [counting the outside of the graph as one of the faces], thus
our planar drawing of $K_5$ must have $7$ faces including the
unbounded face (because $2 = 5 - 10 + F$).

Each edge borders two faces, so $2E = pF$, where $p$ is the average number
of sides on all faces. $K_5$ is a complete graph, so it has no
loops or parallel edges. Because there are no loops, there are no
faces bounded by only one edge, and because there are no parallel
edges there are no faces bounded by two edges. Thus the average number
of edges per face is at least three. So, $p \ge 3$ and $2E \ge 3F$. But $F =
7$ and $E = 10$ implies that $20 \ge 21$, which is a
contradiction. It must be that $K_5$ is non planar.

In a similar way we can prove that the complete bipartite graph
$K_{3,3}$ is not planar (give it a try!). The key difference is
that because $K_{3,3}$ is bipartite, a path that begins and ends
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Your job: complete the proof suggested in the last paragraph.

\textbf{Solution}

Suppose $K_{3,3}$ is a planar graph. Then we can draw
$K_{3,3}$ in the plane so that no edges cross. $K_{3,3}$ has $6$
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\end{enumerate}
\newpage

\verbatiminput{\jobname}

\end{document}
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