We will cover these parts of the book (8th edition):

2.1
2.2.1-2.2.3
2.3
2.4.1, 2.4.2, 2.4.5

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Set Theory

- Set: Collection of objects (“elements/members”)
- $a \in A$ “a is an element of A”
  “a is a member of A”
- $a \notin A$ “a is not an element of A”
- $A = \{a_1, a_2, \ldots, a_n\}$ “A contains…” (roster method)
- Order of elements is meaningless
- It does not matter how often the same element is listed. (generally there are no repetitions)

Examples for Sets

- "Standard" Sets:
- Natural numbers **N** = {0, 1, 2, 3, …}
- Integers **Z** = {…, -2, -1, 0, 1, 2, …}
- Positive Integers $\mathbf{Z}^{+}$ = {1, 2, 3, 4, …}
- Real Numbers **R** = {47.3, -12, π, …}
- Rational Numbers **Q** = {1.5, 2.6, -3.8, 15, …}
- Positive Real Numbers $\mathbf{R}^{+}$
- Complex Numbers **C**
  (correct definitions will follow)

Examples for Sets

- $A = \varnothing$ "empty set/null set"
- $A = \{z\}$ "singleton set" Note: $z \in A$, but $z \neq \{z\}$
- $A = \{\{b, c\}, \{c, x, d\}\}$
- $A = \{\{x, y\}\}$
  Note: $\{x, y\} \in A$, but $\{x, y\} \neq \{\{x, y\}\}$
- $A = \{x \mid x \in \mathbf{N} \wedge x > 7\} = \{8, 9, 10, \ldots\}$
  "set builder notation"
- $[a,b] = \{x \mid x \geq a \wedge x \leq b\}$
  "Closed interval"
- $(a,b) = \{x \mid x > a \wedge x < b\}$
  "Open interval"

Examples for Sets

- We are now able to define the set of rational numbers Q:
- $\mathbf{Q} = \{a/b \mid a \in \mathbf{Z} \wedge b \in \mathbf{Z}^+\}$
- or
- $\mathbf{Q} = \{a/b \mid a \in \mathbf{Z} \wedge b \in \mathbf{Z} \wedge b \neq 0\}$
- And how about the set of real numbers R?
- $\mathbf{R} = \{r \mid r \text{ is a real number}\}$

That is the best we can do.

Subsets

- $A \subseteq B$ "A is a subset of B"
- $A \subseteq B$ if and only if every element of A is also an element of B.
- We can completely formalize this:
- $A \subseteq B \Leftrightarrow \forall x\ (x \in A \rightarrow x \in B)$

- Examples:

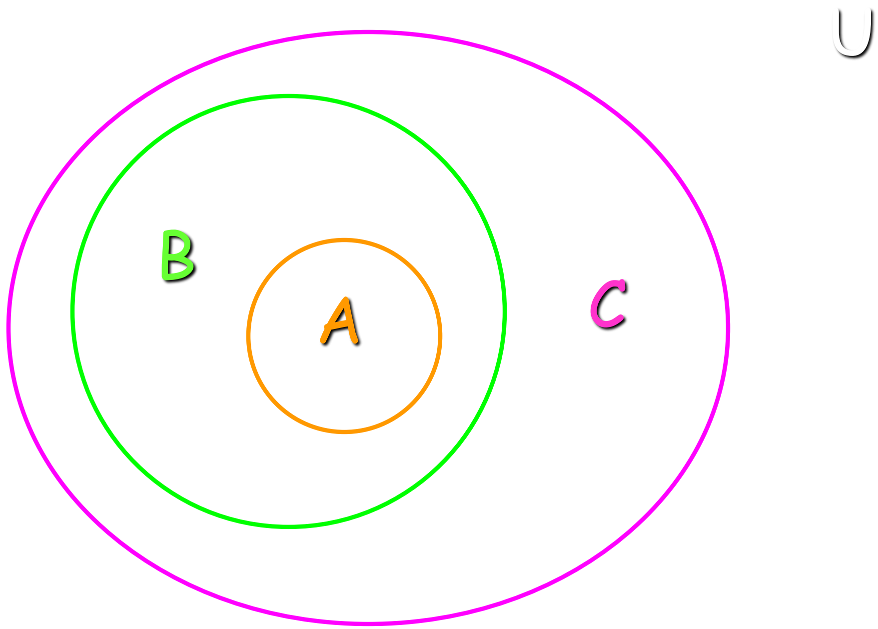
A = {3, 9}, B = {5, 9, 1, 3}, $A \subseteq B$? true

A = {3, 3, 3, 9}, B = {5, 9, 1, 3}, $A \subseteq B$? true

A = {1, 2, 3}, B = {2, 3, 4}, $A \subseteq B$? false

Subsets

- Useful rules:
- $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$
- $(A \subseteq B) \wedge (B \subseteq C) \Rightarrow A \subseteq C$ (see Venn Diagram)



Subsets

- Useful rules:
- $\varnothing \subseteq A$ for any set A
- $A \subseteq A$ for any set A

- Proper subsets:
- $A \subset B$ “A is a proper subset of B”
- $A \subset B \Leftrightarrow \forall x\,(x \in A \rightarrow x \in B) \land \exists x\,(x \in B \land x \notin A)$
- or
- $A \subset B \Leftrightarrow \forall x\,(x \in A \rightarrow x \in B) \land \neg\forall x\,(x \in B \rightarrow x \in A)$

Set Equality

- Sets A and B are equal if and only if they contain exactly the same elements. $\forall x(x \in A \leftrightarrow x \in B)$ or $(A \subseteq B) \wedge (B \subseteq A)$

- Examples:

- A = {9, 2, 7, -3}, B = {7, 9, -3, 2} : A = B

- A = {dog, cat, horse},
  B = {cat, horse, squirrel, dog} : A ≠ B

- A = {dog, cat, horse},
  B = {cat, horse, dog, dog} : A = B

Cardinality of Sets

- If a set S contains n distinct elements, $n \in \mathbf{N}$, we call S a finite set with cardinality n.

- Examples:

A = {Mercedes, BMW, Porsche}, |A| = 3

B = {1, {2, 3}, {4, 5}, 6} |B| = 4

C = Ø |C| = 0

D = { $x \in \mathbf{N}$ | $x \leq 7000$ } |D| = 7001

E = { $x \in \mathbf{N}$ | $x > 7000$ } E is infinite!

The Power Set

- 2^A or $P(A)$ "power set of A"
- $2^A = \{B \mid B \subseteq A\}$ (contains all subsets of A)

- Examples:

- $A = \{x, y, z\}$
- $2^A = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}$

- $A = \emptyset$
- $2^A = \{\emptyset\}$
- Note: $|A| = 0$, $|2^A| = 1$

The Power Set

- Cardinality of power sets:
- $| 2^A | = 2^{|A|}$
- Imagine each element in A has an "**on**/off" switch
- Each possible switch configuration in A corresponds to one element in 2^A

A	1	2	3	4	5	6	7	8
x	x	x	x	x	x	x	x	x
y	y	y	y	y	y	y	y	y
z	z	z	z	z	z	z	z	z

- For 3 elements in A, there are 2x2x2 = 8 elements in 2^A

Cartesian Product

- The ordered n-tuple $(a_1, a_2, a_3, \ldots, a_n)$ is an ordered collection of objects.
- Two ordered n-tuples $(a_1, a_2, a_3, \ldots, a_n)$ and $(b_1, b_2, b_3, \ldots, b_n)$ are equal if and only if they contain exactly the same elements in the same order, i.e., $a_i = b_i$ for $1 \leq i \leq n$.
- The Cartesian product of two sets is defined as:
- $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$
- Example: $A = \{x, y\}$, $B = \{a, b, c\}$

$A \times B = \{(x, a), (x, b), (x, c), (y, a), (y, b), (y, c)\}$

Cartesian Product

- Note that:
- $A \times \emptyset = \emptyset$
- $\emptyset \times A = \emptyset$
- For non-empty sets A and B: $A \neq B \Leftrightarrow A \times B \neq B \times A$
- $|A \times B| = |A| \cdot |B|$

- The Cartesian product of two or more sets is defined as:
- $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i \text{ for } 1 \leq i \leq n\}$

Partitions

- **Definition:** A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, $i \in I$, forms a partition of S if and only if

-

(i) $A_i \neq \emptyset$ for $i \in I$

(ii) $A_i \cap A_j = \emptyset$, if $i \neq j$

(iii) $\cup_{i \in I} A_i = S$

Partitions

- **Examples:** Let S be the set {u, m, b, r, o, c, k, s}. Do the following collections of sets partition S ?

{{m, o, c, k}, {r, u, b, s}}	yes.
{{c, o, m, b}, {u, s}, {r}}	no (k is missing).
{{b, r, o, c, k}, {m, u, s, t}}	no (t is not in S).
{{u, m, b, r, o, c, k, s}}	yes.
{{b, o, r, k}, {r, u, m}, {c, s}}	no (r is in two sets).
{{u, m, b}, {r, o, c, k, s}, ∅}	no (∅ not allowed).

Set Operations

- Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$
- Example: $A = \{a, b\}$, $B = \{b, c, d\}$
- $A \cup B = \{a, b, c, d\}$
- $|A \cup B| = |A| + |B| - |A \cap B|$
- Intersection: $A \cap B = \{x \mid x \in A \wedge x \in B\}$
- Example: $A = \{a, b\}$, $B = \{b, c, d\}$
- $A \cap B = \{b\}$

Set Operations

- Two sets are called disjoint if their intersection is empty, that is, they share no elements:
- $A \cap B = \emptyset$

- The difference between two sets A and B contains exactly those elements of A that are not in B:
- $A\text{-}B = \{x \mid x \in A \wedge x \notin B\}$

Example: A = {a, b}, B = {b, c, d}, A-B = {a}

Set Operations

- The complement of a set A contains exactly those elements under consideration that are not in A:
- -A = U-A

- Example: U = **N**, B = {250, 251, 252, …}
- -B = {0, 1, 2, …, 248, 249}

- $A - B = A \cap \bar{B}$

Set Operations

- How can we prove $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$?
- Method I:
- $x \in A \cup (B \cap C)$

$\Leftrightarrow x \in A \vee x \in (B \cap C)$

$\Leftrightarrow x \in A \vee (x \in B \wedge x \in C)$

$\Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)$
(distributive law for logical expressions)

$\Leftrightarrow x \in (A \cup B) \wedge x \in (A \cup C)$

$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$

Set Operations

- Method II: Membership table
- 1 means “x is an element of this set”

0 means “x is not an element of this set”

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
0	0	0	0	0	0	0	0
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Set Identities

Identity	Name
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\bar{A})} = A$	Complementation laws
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

Identity	Name
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{(A \cap B)} = \bar{A} \cup \bar{B}$ $\overline{(A \cup B)} = \bar{A} \cap \bar{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \bar{A} = U$ $A \cap \bar{A} = \emptyset$	Complement laws

Proving Set Identities

Description	Method
Subset method	Show that each side of the identity is a subset of the other side.
Membership table	For each possible combination of the atomic sets, show that an element in exactly these atomic sets must either belong to both sides or belong to neither side.
Apply existing identities	Start with one side, transform it into the other side using a sequence of steps by applying an established identity.

Exercises

- **Question 1:**
- Given a set $A = \{x, y, z\}$ and a set $B = \{1, 2, 3, 4\}$, what is the value of $| 2^A \times 2^B |$?

- **Question 2:**
- Is it true for all sets A and B that $(A \times B) \cap (B \times A) = \emptyset$? Or do A and B have to meet certain conditions?

- **Question 3:**
- For any two sets A and B, if $A - B = \emptyset$ and $B - A = \emptyset$, can we conclude that $A = B$? Why or why not?

Exercises

- **Question 1:**
- Given a set $A = \{x, y, z\}$ and a set $B = \{1, 2, 3, 4\}$, what is the value of $| 2^A \times 2^B |$?

- **Answer:**
- $| 2^A \times 2^B | = | 2^A | \cdot | 2^B | = 2^{|A|} \cdot 2^{|B|} = 8 \cdot 16 = 128$

Exercises

- **Question 2:**
- Is it true for all sets A and B that $(A \times B) \cap (B \times A) = \emptyset$? Or do A and B have to meet certain conditions?

- **Answer:**
- If A and B share at least one element x, then both $(A \times B)$ and $(B \times A)$ contain the pair (x, x) and thus are not disjoint.
- Therefore, for the above equation to be true, it is necessary that $A \cap B = \emptyset$.

Exercises

- **Question 3:**
- For any two sets A and B, if $A - B = \emptyset$ and $B - A = \emptyset$, can we conclude that $A = B$? Why or why not?

- **Answer:**
- Proof by contradiction: Assume that $A \neq B$.
- Then there must be either an element x such that $x \in A$ and $x \notin B$ or an element y such that $y \in B$ and $y \notin A$.
- If x exists, then $x \in (A - B)$, and thus $A - B \neq \emptyset$.
- If y exists, then $y \in (B - A)$, and thus $B - A \neq \emptyset$.
- This contradicts the premise $A - B = \emptyset$ and $B - A = \emptyset$, and therefore we can conclude $A = B$.

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… and the next section is about…

- Functions

Functions

- A function f from a set A to a set B is an assignment of exactly one element of B to each element of A.
- We write
- $f(a) = b$
- if b is the unique element of B assigned by the function f to the element a of A.
- If f is a function from A to B, we write
- $f: A \to B$
- (note: Here, "$\to$" has nothing to do with if… then)

Functions

- If f:A→B, we say that A is the domain of f and B is the codomain of f.
- If f(a) = b, we say that b is the image of a and a is the pre-image of b.
- The range of f:A→B is the set of all images of elements of A.
- We say that f:A→B maps A to B.

Functions

- Let us take a look at the function $f:P \rightarrow C$ with
- P = {Linda, Max, Kathy, Peter}
- C = {Boston, New York, Hong Kong, Moscow}

- f(Linda) = Moscow
- f(Max) = Boston
- f(Kathy) = Hong Kong
- f(Peter) = New York

- Here, the range of f is C.

Functions

- Let us re-specify f as follows:

- f(Linda) = Moscow
- f(Max) = Boston
- f(Kathy) = Hong Kong
- f(Peter) = Boston

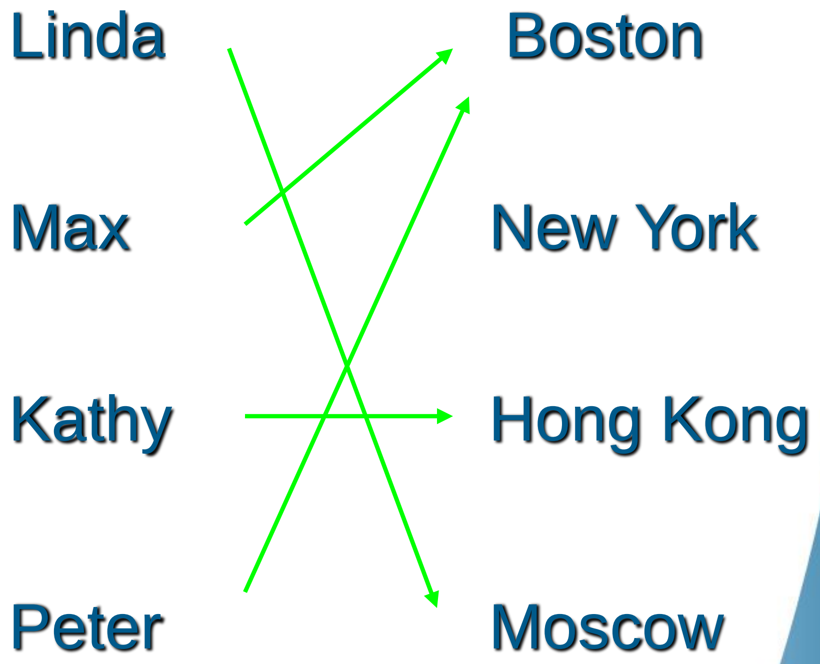
- Is f still a function? yes

What is its range? {Moscow, Boston, Hong Kong}

Functions

- Other ways to represent f:

x	f(x)
Linda	Moscow
Max	Boston
Kathy	Hong Kong
Peter	Boston



Functions

- If the domain of our function f is large, it is convenient to specify f with a formula, e.g.:
- f:**R**→**R**
- f(x) = 2x
- This leads to:
- f(1) = 2
- f(3) = 6
- f(-3) = -6
- …

Functions

- Let f_1 and f_2 be functions from A to **R**.
- Then the sum and the product of f_1 and f_2 are also functions from A to **R** defined by:
- $(f_1 + f_2)(x) = f_1(x) + f_2(x)$
- $(f_1f_2)(x) = f_1(x) f_2(x)$

- Example:
- $f_1(x) = 3x, \; f_2(x) = x + 5$
- $(f_1 + f_2)(x) = f_1(x) + f_2(x) = 3x + x + 5 = 4x + 5$
- $(f_1f_2)(x) = f_1(x) f_2(x) = 3x (x + 5) = 3x^2 + 15x$

Functions

- We already know that the range of a function $f:A\rightarrow B$ is the set of all images of elements $a\in A$.

- If we only regard a subset $S\subseteq A$, the set of all images of elements $s\in S$ is called the image of S.

- We denote the image of S by f(S):

- $f(S) = \{f(s) \mid s\in S\}$

Functions

- Let us look at the following well-known function:
- f(Linda) = Moscow
- f(Max) = Boston
- f(Kathy) = Hong Kong
- f(Peter) = Boston
- What is the image of S = {Linda, Max} ?
- f(S) = {Moscow, Boston}
- What is the image of S = {Max, Peter} ?
- f(S) = {Boston}

Properties of Functions

- A function $f:A \rightarrow B$ is said to be one-to-one (or injective), if and only if
- $\forall x, y \in A \, (f(x) = f(y) \rightarrow x = y)$
- In other words: f is one-to-one if and only if it does not map two distinct elements of A onto the same element of B.
- Note that a function f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$. This way of expressing that f is one-to-one is obtained by taking the contrapositive of the implication in the definition.

Properties of Functions

- And again…
- f(Linda) = Moscow
- f(Max) = Boston
- f(Kathy) = Hong Kong
- f(Peter) = Boston

- Is f one-to-one?

- No, Max and Peter are mapped onto the same element of the image.

g(Linda) = Moscow
g(Max) = Boston
g(Kathy) = Hong Kong
g(Peter) = New York

Is g one-to-one?

Yes, each element is assigned a unique element of the image.

Properties of Functions

- How can we prove that a function f is one-to-one?
- Whenever you want to prove something, first take a look at the relevant definition(s):
- $\forall x, y \in A \ (f(x) = f(y) \rightarrow x = y)$
- Example:
- f:**R**→**R**
- $f(x) = x^2$
- Disproof by counterexample:
- $f(3) = f(-3)$, but $3 \neq -3$, so f is not one-to-one.

Properties of Functions

- … and yet another example:
- f:**R**→**R**
- f(x) = 3x
- One-to-one: $\forall x, y \in A\ (f(x) = f(y) \to x = y)$
- To show: $f(x) \neq f(y)$ whenever $x \neq y$
- $x \neq y$

$\Leftrightarrow 3x \neq 3y$

$\Leftrightarrow f(x) \neq f(y)$,

so if $x \neq y$, then $f(x) \neq f(y)$, that is, f is one-to-one.

Properties of Functions

- A function f:A→B with A,B $\subseteq$ **R** is called increasing, if $\forall x, y \in A\ (x < y \rightarrow f(x) \leq f(y))$, and strictly increasing, if $\forall x, y \in A\ \left(x < y \rightarrow f(x) < f(y)\right)$.
- f is decreasing if $\forall x, y \in A\ (x < y \rightarrow f(x) \geq f(y))$, and strictly decreasing if
- $\forall x, y \in A\ (x < y \rightarrow f(x) > f(y))$

- Obviously, a function that is either strictly increasing or strictly decreasing is one-to-one.

Properties of Functions

- A function $f: A \to B$ is called onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

- In other words, f is onto if and only if its range is its entire codomain.

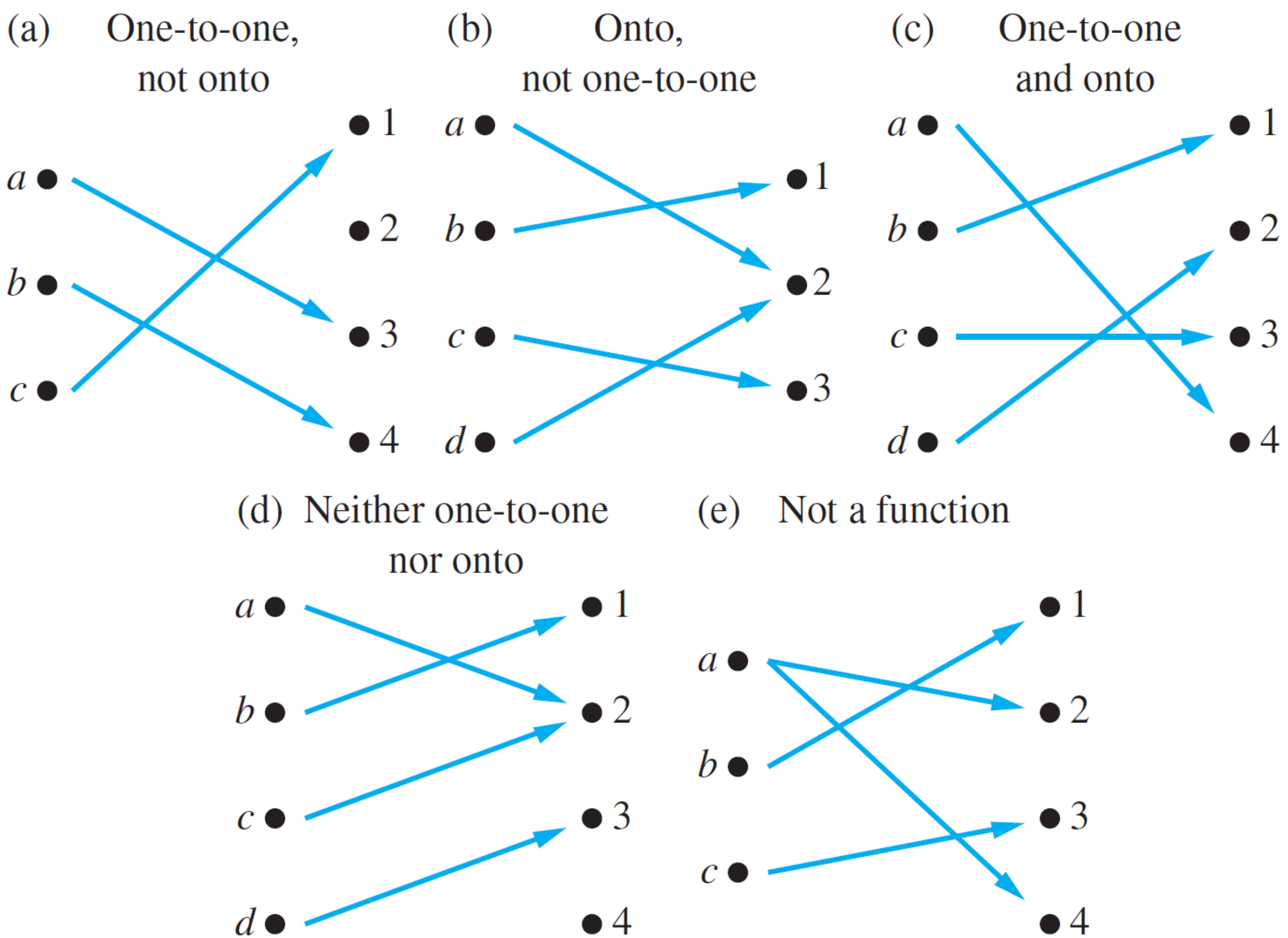
- A function $f: A \to B$ is a one-to-one correspondence, or a bijection, if and only if it is both one-to-one and onto.

- Obviously, if f is a bijection and A and B are finite sets, then $|A| = |B|$.

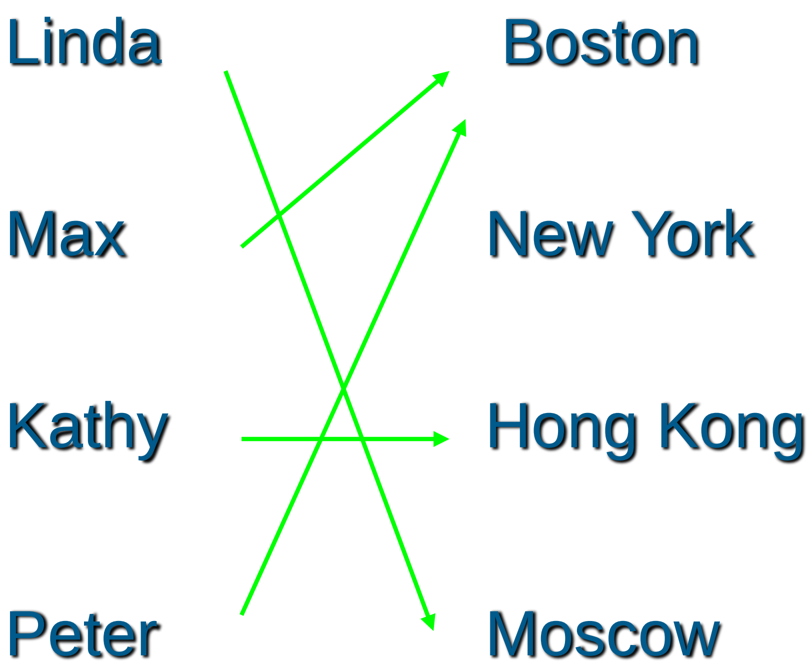
Properties of Functions

- Examples:
- In the following examples, we use the arrow representation to illustrate functions f:A→B.
- In each example, the complete sets A and B are shown.

Properties of Functions



Properties of Functions

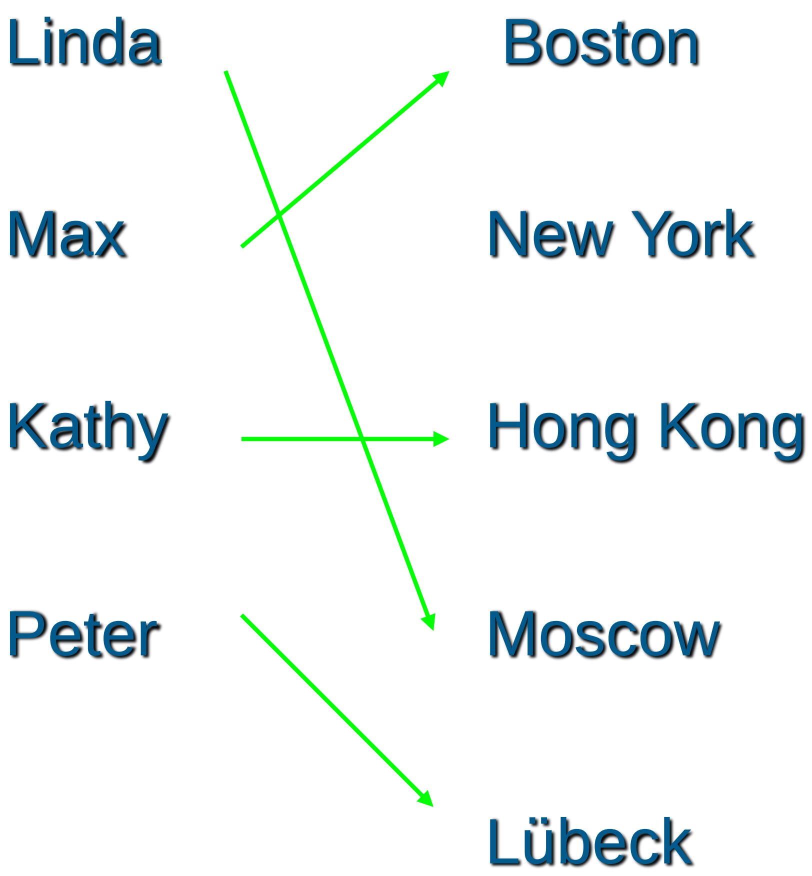


- Is f injective?
- No.
- Is f surjective?
- No.
- Is f bijective?
- No.

Properties of Functions

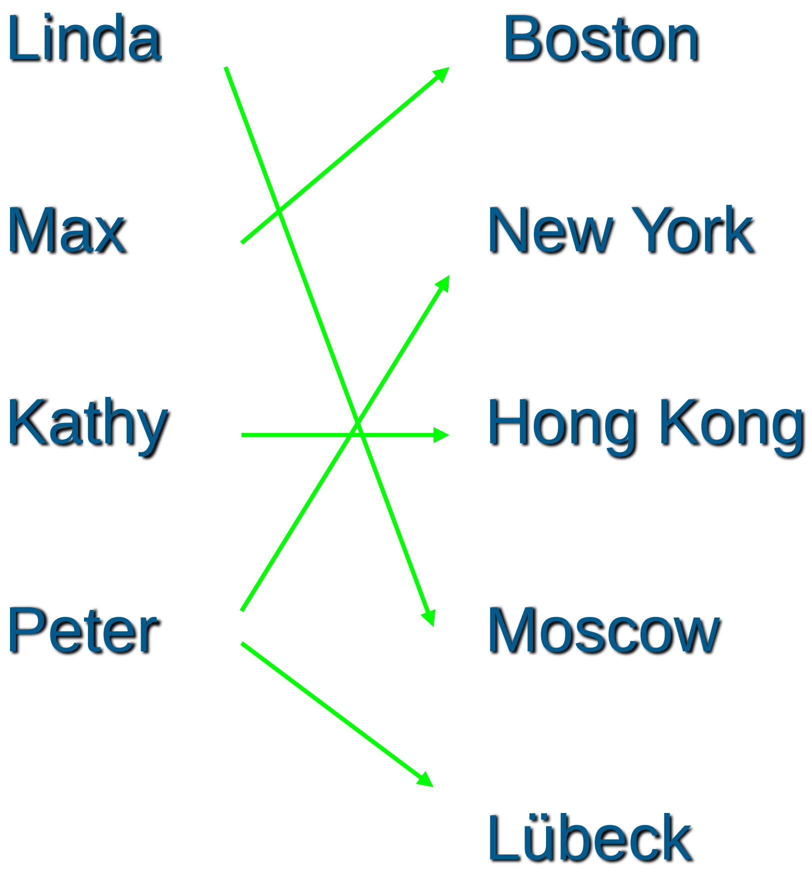
- Is f injective?
- No.
- Is f surjective?
- Yes.
- Is f bijective?
- No.

Properties of Functions



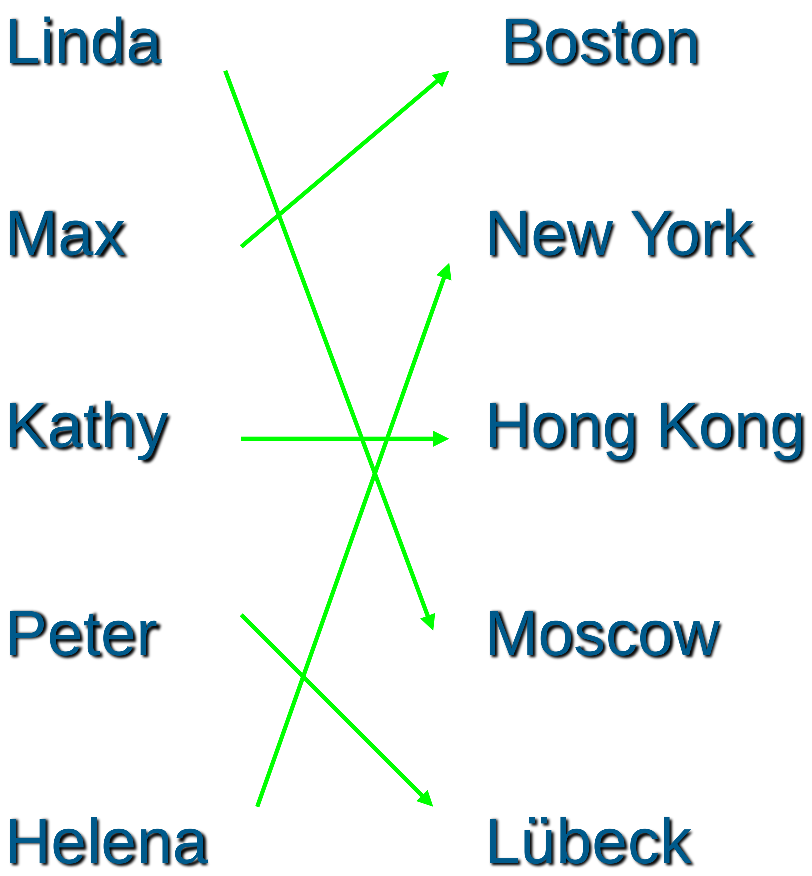
- Is f injective?
- Yes.
- Is f surjective?
- No.
- Is f bijective?
- No.

Properties of Functions



- Is f injective?
- No! f is not even a function!

Properties of Functions



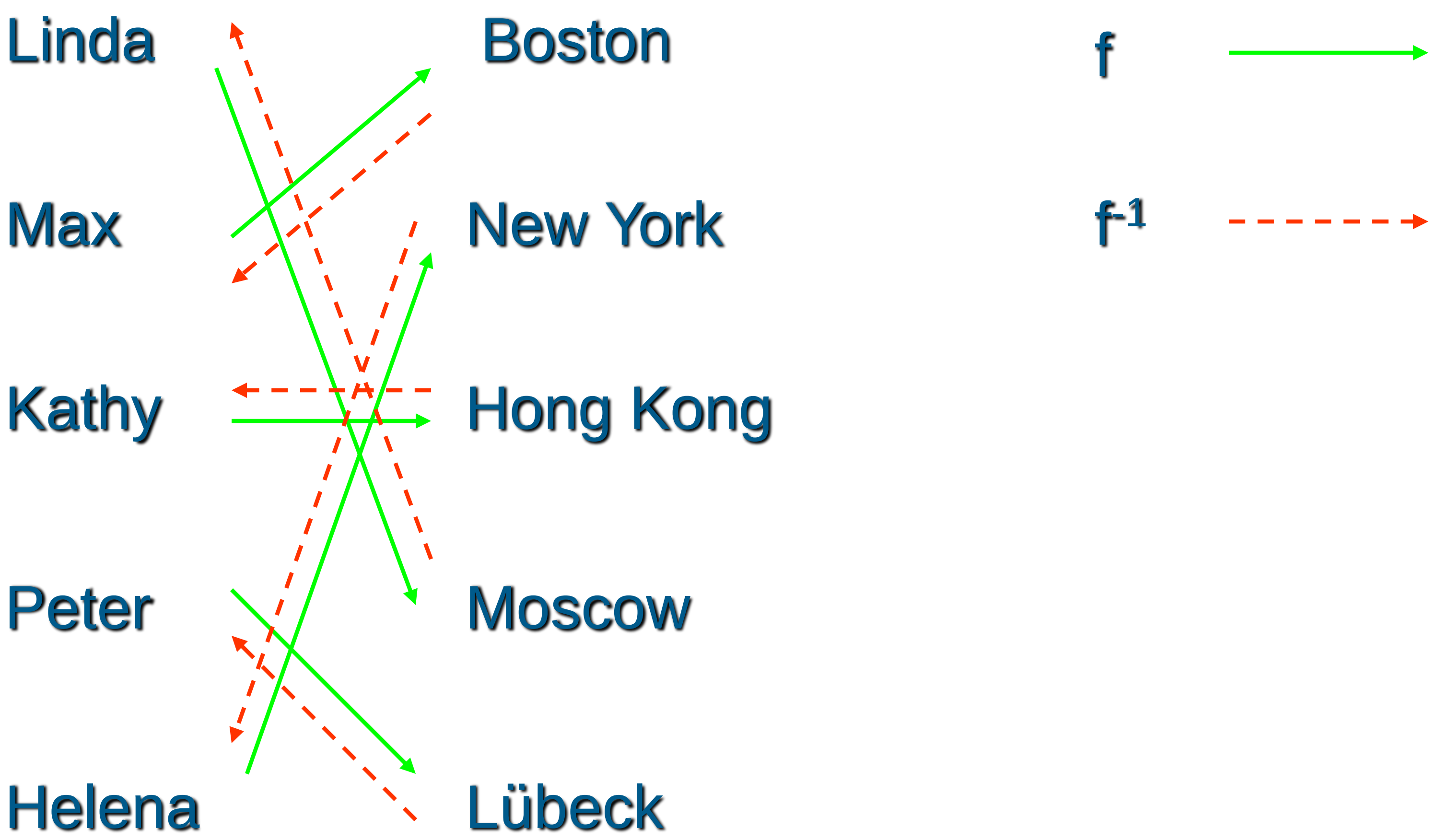
- Is f injective?
- Yes.
- Is f surjective?
- Yes.
- Is f bijective?
- Yes.

Inversion

- An interesting property of bijections is that they have an inverse function.

- The inverse function of the bijection $f:A \rightarrow B$ is the function $f^{-1}:B \rightarrow A$ with
- $f^{-1}(b) = a$ whenever $f(a) = b$.

Inversion


Linda
Max
Kathy
Peter
Helena
Boston
New York
Hong Kong
Moscow
Lübeck
f
f-1

Inversion

Example:

f(Linda) = Moscow
f(Max) = Boston
f(Kathy) = Hong Kong
f(Peter) = Lübeck
f(Helena) = New York

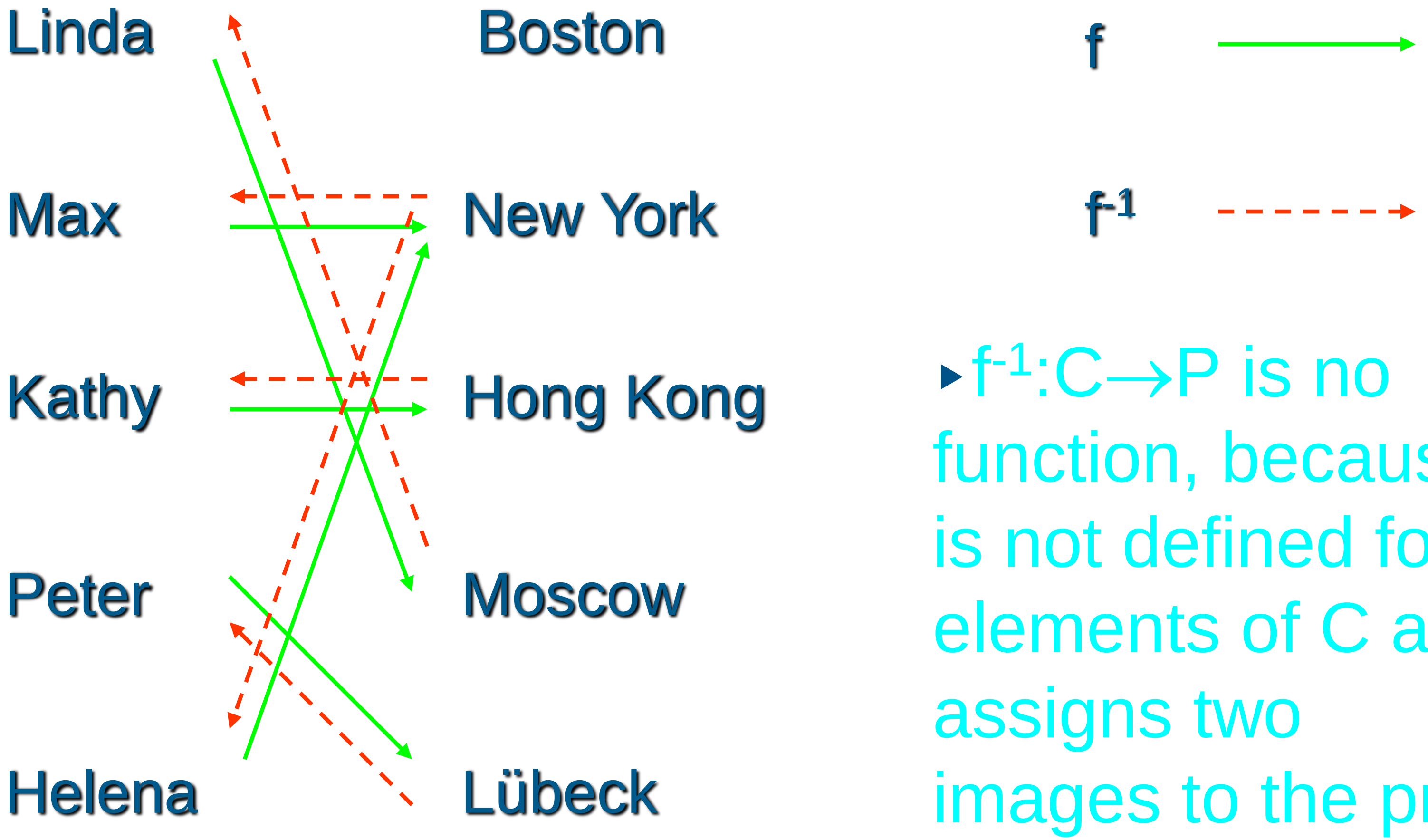
Clearly, f is bijective.

The inverse function f^{-1} is given by:

f^{-1}(Moscow) = Linda
f^{-1}(Boston) = Max
f^{-1}(Hong Kong) = Kathy
f^{-1}(Lübeck) = Peter
f^{-1}(New York) = Helena

Inversion is only possible for bijections
(= invertible functions)

Inversion



- $f^{-1}: C \rightarrow P$ is no function, because it is not defined for all elements of C and assigns two images to the pre-image New York.

Composition

- The **composition** of two functions $g:A \to B$ and $f:B \to C$, denoted by $f \circ g$, is defined by
- $(f \circ g)(a) = f(g(a))$
- This means that
  - **first**, function g is applied to element $a \in A$, mapping it onto an element of B,
  - **then**, function f is applied to this element of B, mapping it onto an element of C.
  - **Therefore**, the composite function maps from A to C.

Composition

- Example:

- $f(x) = 7x – 4$, $g(x) = 3x$,
- $f:\mathbf{R}\rightarrow\mathbf{R}$, $g:\mathbf{R}\rightarrow\mathbf{R}$

- $(f\circ g)(5) = f(g(5)) = f(15) = 105 – 4 = 101$

- $(f\circ g)(x) = f(g(x)) = f(3x) = 21x - 4$

Composition

- Composition of a function and its inverse:
- $(f^{-1} \circ f)(x) = f^{-1}(f(x)) = x$
- The composition of a function and its inverse is the **identity function** $i(x) = x$.

Graphs

- The **graph** of a function f:A→B is the set of ordered pairs {(a, b) | a∈A and f(a) = b}.

- The graph is a subset of A×B that can be used to visualize f in a two-dimensional coordinate system.

- From the definition, the graph of a function f from A to B is the subset of $A \times B$ containing the ordered pairs with the second entry equal to the element of B assigned by f to the first entry.

Floor and Ceiling Functions

- The **floor** and **ceiling** functions map the real numbers onto the integers (**R**→**Z**).
- The **floor** function assigns to r∈**R** the largest z∈**Z** with z≤r, denoted by $\lfloor r \rfloor$.
- **Examples:** $\lfloor 2.3 \rfloor = 2$, $\lfloor 2 \rfloor = 2$, $\lfloor 0.5 \rfloor = 0$, $\lfloor -3.5 \rfloor = -4$
- The **ceiling** function assigns to r∈**R** the smallest z∈**Z** with z≥r, denoted by $\lceil r \rceil$.
- **Examples:** $\lceil 2.3 \rceil = 3$, $\lceil 2 \rceil = 2$, $\lceil 0.5 \rceil = 1$, $\lceil -3.5 \rceil = -3$

Floor and Ceiling Functions

- Useful properties of the Floor and Ceiling functions

(n is an integer and x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \leq x < n + 1$
(1b)	$\lceil x \rceil = n$ if and only if $n - 1 < x \leq n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x - 1 < n \leq x$
(1d)	$\lceil x \rceil = n$ if and only if $x \leq n < x + 1$
(2)	$x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

Sequences

- **Sequences** represent **ordered lists** of elements.
- A **sequence** is defined as a function from a subset of **N** to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a term of the sequence.
- **Example:**
- subset of **N**: 1 2 3 4 5 …

↓ ↓ ↓ ↓ ↓

S: 2 4 6 8 10 …

Sequences

- We use the notation $\{a_n\}$ to describe a sequence.
- Important: Do not confuse this with the {} used in set notation.
- It is convenient to describe a sequence with an **equation**.
- For example, the sequence on the previous slide can be specified as $\{a_n\}$, where $a_n = 2n$.

The Equation Game

What are the equations that describe the following sequences $a_1, a_2, a_3, \ldots$?

- 1, 3, 5, 7, 9, … $a_n = 2n - 1$

-1, 1, -1, 1, -1, … $a_n = (-1)^n$

2, 5, 10, 17, 26, … $a_n = n^2 + 1$

0.25, 0.5, 0.75, 1, 1.25 … $a_n = 0.25n$

3, 9, 27, 81, 243, … $a_n = 3^n$

Strings

- Finite sequences are also called **strings**, denoted by $a_1a_2a_3\ldots a_n$.

- The **length** of a string S is the number of terms that it consists of.

- The **empty string** contains no terms at all. It has length zero.

Summations

- What does $\sum_{j=m}^{n} a_j$ stand for?

- It represents the sum $a_m + a_{m+1} + a_{m+2} + \ldots + a_n$.

- The variable j is called the **index of summation**, running from its **lower limit** m to its **upper limit** n. We could as well have used any other letter to denote this index.

Geometric and Arithmetic progressions

- The sequence $a, ar, ar^2, \ldots, ar^n, \ldots$ is a geometric progression where the initial term a and the common ratio r are real numbers.

- The sequence $a, a + d, a + 2d, \ldots, a + nd, \ldots$ is an arithmetic progression where the initial term a and the common difference d are real numbers.

Some useful Summation Formulae

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Summations

How can we express the sum of the first 1000 terms of the sequence $\{a_n\}$ with $a_n=n^2$ for
n = 1, 2, 3, … ?

We write it as $\sum_{j=1}^{1000} j^2$

What is the value of $\sum_{j=1}^{6} j$?

- It is 1 + 2 + 3 + 4 + 5 + 6 = 21.

What is the value of $\sum_{j=1}^{100} j$?

It is so much work to calculate this…

Summations

- It is said that Carl Friedrich Gauss came up with the following formula:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

When you have such a formula, the result of any summation can be calculated much more easily, for example:

$$\sum_{j=1}^{100} j = \frac{100(100+1)}{2} = \frac{10100}{2} = 5050$$

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Double Summations

- Corresponding to nested loops in C or Java, there is also double (or triple etc.) summation. To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation:
- Example:
- $\sum_{i=1}^{4} \sum_{j=1}^{3} ij = \sum_{i=1}^{4} (i + 2i$