HW3: Number Theory, Induction, Recursion, Counting, Discrete Probability

Assigned: 26 July 2020 **Due:** 9 August 2020

1. **Exercise 44 page 259**

n is an integer so it's even or odd.

a) $n = 2k \rightarrow n^2 = 4k^2 \rightarrow n^2 \equiv 0\ (mod\ 4)$

b) $n = 2k + 1 \rightarrow n^2 = 4k^2 + 4k + 1 = 4k' + 1 \rightarrow n^2 \equiv 1\ (mod\ 4)$

2. **Exercise 28 page 269**

Binary expansion of 1001 = $(1111101001)_2$ and $power = 123\ \ mod\ 101 = 22$

i=0: Because $a_0 = 1$, we have $x = 1 \cdot 22\, mod\ 101 = 22$ and $power = 22^2\ mod\ 101 = 80$

i=1: Because $a_1 = 0$, we have $x = 22$ and $power = 22^2\ mod\ 101 = 80$

i=2: Because $a_2 = 0$, we have $x = 22$ and $power = 80^2\ mod\ 101 = 37$

i=3: Because $a_3 = 1$, we have $x = 22 \cdot 37\, mod\ 101 = 6$ and $power = 37^2\ mod\ 101 = 56$

i=4: Because $a_4 = 0$, we have $x = 6$ and $power = 56^2\ mod\ 101 = 5$

i=5: Because $a_5 = 1$, we have $x = 6 \cdot 5\, mod\ 101 = 30$ and $power = 5^2\ mod\ 101 = 25$

i=6: Because $a_6 = 1$, we have $x = 30 \cdot 25\, mod\ 101 = 43$ and $power = 25^2\ mod\ 101 = 19$

i=7: Because $a_7 = 1$, we have $x = 43 \cdot 19\, mod\ 101 = 9$ and $power = 19^2\ mod\ 101 = 58$

i=8: Because $a_8 = 1$, we have $x = 9 \cdot 58\, mod\ 101 = 17$ and $power = 58^2\ mod\ 101 = 31$

i=9: Because $a_9 = 1$, we have $x = 17 \cdot 31\, mod\ 101 = 22$

So $123^{1001}\, mod\ 101 = 22$

3. **Exercise 42 page 290**

$356 = 252 \cdot 1 + 104$

$252 = 104 \cdot 2 + 44$

$104 = 44 \cdot 2 + 16$

$44 = 16 \cdot 2 + 12$

$16 = 12 \cdot 1 + 4$

$12 = 4 \cdot 3 + 0$

So $q_1 = 1,\ q_2 = 2,\ q_3 = 2,\ q_4 = 2,\ q_5 = 1,\ q_6 = 3$

Set $s_0 = 1,\ s_1 = 0,\ t_0 = 0,\ t_1 = 1$. And $s_j = s_{j-2} - q_{j-1}s_{j-1}$, and $t_j = t_{j-2} - q_{j-1}t_{j-1}$ for $j = 2,3,4,5,6$.

$s_2 = 1, s_3 = -2, s_4 = 5, s_5 = -12, s_6 = 17$

$t_2 = -1, t_3 = 3, t_4 = -7, t_5 = 17, t_6 = -24$

So $\gcd(356,252) = 17 \cdot 356 - 24 \cdot 252 = 4$

4. **Exercise 18 page 351**

a) $2! < 2^2$

b) It's true. Because $2 < 4$

c) $k! < k^k$

d) For each $k \geq 2$, $P(k)$ implies $P(k+1)$. So we want to show that assuming the inductive hypothesis, we can show that $(k+1)! < (k+1)^{(k+1)}$.

e) $(k+1)! = (k+1) \times k! < (k+1) \times k^k < (k+1)^k < (k+1)^{(k+1)}$

f) We have completed both the basis step and the inductive step, so by the principle of mathematical induction, the statement is true for every integer n greater than 1.

5. **Exercise 6 page 363**

a) 3, 6, 9, 10, 12, 13, 15, 16, and all the number greater or equal to 18.

b) Let $P(n)$ be the statement that we can form n cents of postage using just 3-cent and 10-cent stamps. We want to prove that $P(n)$ is true for all $n \geq 18$. For the basis step, $18 = 3 + 3 + 3 + 3 + 3 + 3$. For the inductive hypothesis assume that we can form k cents of postage. We will show how to form $k + 1$ cents of postage. If the k cents included two 10-cent stamps, replace those two 10-cent stamps with seven 3-cent stamps. Otherwise, if they included only one 10-cent stamp, it should include at least three 3-cent stamps as well (Because we are talking about numbers greater than or equal to 18). Replace three 3-cent stamps with a 10-cent stamp. And finally, if they don't include any 10-cent stamps, it should include at least six 3-cent stamps. Replace three 3-cent stamps with a 10-cent stamp. So we have formed $k + 1$ cents in postage.

c) Here $P(n)$ is same as part b. To prove that $P(n)$ is true for all $n \geq 18$, we check for the basis step that $18 = 3 + 3 + 3 + 3 + 3 + 3$, $19 = 10 + 3 + 3 + 3$, and $20 = 10 + 10$. For the inductive step, assume that the inductive hypothesis, that $P(j)$ is true for all j with $18 \leq j \leq k$, where k is an arbitrary integer greater than or equal to 20. We want to show that $P(k + 1)$ is true. Because $k - 2 \geq 18$, we know that $P(k - 2)$ is true, that is, that we can form $k - 2$ cents pf postage. Add one more 3-cent stamp and we have formed $k + 1$ cents of postage. So we used all the values between k and 18.

6. Exercise 8 page 379

a) $a_{n+1} = a_n + 4$ for $n \geq 1$ and $a_1 = 2$.

b) $a_{n+2} = a_n$ for $n \geq 1$ and $a_1 = 0$, $a_2 = 2$

c) $a_{n+1} = \frac{n+2}{n} \times a_n$ and $a_1 = 2$

d) $a_{n+1} = a_n + 2\sqrt{a_n} + 1$ and $a_1 = 1$

7. Exercise 8 page 391

procedure $sum\_pos(n$: positive integer)

if $n = 1$ **then return** 1

else return $sum\_pos(n - 1) + n$

Proof:

We use strong induction on n. Basis step: $n = 1$ and we know that $1 = 1$. Inductive step: Fix $k > 1$, assume the inductive hypothesis (that the algorithm works correctly for all values of n less than k) and consider what happens with input k. Because $k > 1$, the **else** clause is executed, and the answer is $\sum_{i=1}^{k-1} i + k = \sum_{i=1}^{k} i$ which equals the sum of the first n positive integers.

8. Exercise 26 page 417

a) (4 different digits) or (1 digit for three positions and another digit for the other position) or (1 digit for all positions) = $\Big(\big(C(10,1)C(9,1)C(8,1)C(7,1)\big) + \big(C(10,1)C(9,1)C(4,1)\big) + \big(C(10,1)\big)\Big) = 5410$

b) $C(10,1)C(10,1)C(10,1)C(5,1) = 5000$

c) $C(4,1)C(9,1) = 36$

9. Exercise 46 page 428

There are exactly 50 even numbers and 50 odd numbers between 1000 and 1099 so there are at most 50 not consecutive integers between them. So the pigeonhole principle shows that at least two houses have addresses that are consecutive integers.

10. Exercise 34 page 436

a) $\bar{A} = U - A \rightarrow 26^6 - 25^6 = 64775151$

b) $\overline{A \cap B} = \bar{A} \cup \bar{B} = (U - A) \cup (U - B) \rightarrow (26^6 - 25^6) + (26^6 - 25^6) - (26^6 - 24^6) = 129550302 - 117812800 = 11737502$

c) $C(5,1) \times 24 \times 23 \times 22 \times 21 = 1275120$

d) $C(6,2) \times 24 \times 23 \times 22 \times 21 = 3825360$

11. Exercise 26 page 444

a) Suppose we want to choose two sets of k and r-k elements from n elements. We can do it in 2 ways. First, choose r elements and then choose k elements from those r elements. Second, choose k elements from n elements and then choose r-k elements from the remaining n-k elements.

b) $\binom{n}{r}\binom{r}{k} = \frac{n!}{r!\times(n-r)!} \times \frac{r!}{k!\times(r-k)!} = \frac{n!}{(n-r)!} \times \frac{1}{k!\times(r-k)!} = \frac{n!}{(n-r)!\times(n-k)!} \times \frac{(n-k)!}{k!\times(r-k)!} = \frac{n!}{k!\times(n-k)!} \times \frac{(n-k)!}{(n-r)!\times(r-k)!} = \binom{n}{k}\binom{n-k}{r-k}$

12. Exercise 40 page 476

a) $P(E) = \frac{|E|}{|S|} = \frac{1}{C(69,5)\times C(26,1)} = 3.4e-9$

b) $P(E) = \frac{|E|}{|S|} = \frac{C(25,1)}{C(69,5)\times C(26,1)} = 8.6e-8$

c) $P(E) = \frac{|E|}{|S|} = \frac{C(5,3)C(64,2)+C(5,4)C(64,1)C(25,1)}{C(69,5)\times C(26,1)} = \frac{20160+8000}{C(69,5)\times C(26,1)} = 9.6e-5$

d) $P(E) = \frac{|E|}{|S|} = \frac{C(5,1)C(64,4)+C(64,5)}{C(69,5)\times C(26,1)} = 0.037$

13. Exercise 34 page 493

a) $C(n,k)p^k q^{n-k} = C(n,0)p^0(1-p)^n = (1-p)^n$

b) $\overline{a)} = 1-(1-p)^n$

c) $a) + \ C(n,1)p(1-p)^{n-1} = (1-p)^n(1+\frac{np}{1-p})$

d) $\overline{c)} = 1-(1-p)^n(1+\frac{np}{1-p})$

14. Exercise 12 page 518

a) $P(X=n) = \overline{P(6)}^{n-1}P(6) = \left(\frac{5}{6}\right)^{n-1} \times \frac{1}{6}$

b) $E(N) = \frac{1}{P(N)} = \frac{1}{P(6)} = \frac{1}{\frac{1}{6}} = 6$