HW4: Advanced Counting, Relations, Graphs

Assigned: 9 August 2020 **Due:** 16 August 2020

1. **Exercise 2 page 536**
   a) Suppose P(n) is the number of permutations of a set with n elements. We know that P(1) is 1. Because there exists only one permutation for 1 element. Now lets find P(n+1). The first element of this permutation can be any element of the (n+1) elements, so we have (n+1) options for the first element. Then we have n remaining elements for the next positions. So there are P(n) permutations for the rest of positions. Therefore,
   $P(n+1) = (n+1) \times P(n) \rightarrow P(n) = n \times P(n-1)$
   b) $P(n) = nP(n-1) = n(n-1)P(n-2) = \cdots = n(n-1)\ldots 3P(2) = n(n-1)\ldots(3)(2)P(1) = n!$
2. **Exercise 4 page 551**
   a) $r^2 - r - 6 = 0 \rightarrow r = 3, -2 \rightarrow a_n = \alpha_1 3^n + \alpha_2(-2)^n$. We know that $a_0 = 3, a_1 = 6 \rightarrow 3 = \alpha_1 + \alpha_2 \wedge 6 = 3\alpha_1 - 2\alpha_2 \rightarrow \alpha_1 = \frac{12}{5}, \alpha_2 = \frac{3}{5} \rightarrow a_n = \frac{12}{5}3^n + \frac{3}{5}(-2)^n$
   b) $r^2 - 7r + 10 = 0 \rightarrow r = 5, 2 \rightarrow a_n = \alpha_1 5^n + \alpha_2 2^n$. We know that $a_0 = 2, a_1 = 1 \rightarrow 2 = \alpha_1 + \alpha_2 \wedge 1 = 5\alpha_1 + 2\alpha_2 \rightarrow \alpha_1 = -1, \alpha_2 = 3 \rightarrow a_n = -5^n + 3(2)^n$
   c) $r^2 - 6r + 8 = 0 \rightarrow r = 4, 2 \rightarrow a_n = \alpha_1 4^n + \alpha_2 2^n$. We know that $a_0 = 4, a_1 = 10 \rightarrow 4 = \alpha_1 + \alpha_2 \wedge 10 = 4\alpha_1 + 2\alpha_2 \rightarrow \alpha_1 = 1, \alpha_2 = 3 \rightarrow a_n = 4^n + 3(2)^n$
3. **Exercise 22 page 562**
   a) $f(16) = 2f(4) + \log 16 = 2(2f(2) + \log 4) + 4\log 2 = 4 + 2\log 4 + 4\log 2 = 4 + 8\log 2 = 12$

b) $m = \log n \rightarrow 2^m = n \rightarrow f(2^m) = 2f\left(\sqrt{2^m}\right) + m$. Suppose $g(m) = f(2^m) \rightarrow g(m) = 2g\left(\frac{m}{2}\right) + m$. Now we can use Master theorem: $a = 2, b = 2, c = 1, d = 1 \rightarrow g(m) = O(m \log m) \rightarrow f(n) = O(\log n \log \log n)$

4. Exercise 6 page 608

b) It's reflexive because $x = x$ for any $x \in R$. It's symmetric because if $x = \pm y \rightarrow y = \pm x$. It's not antisymmetric because $(-2,2) \in R, (2,-2) \in R$, but $-2 \neq 2$. It's transitive because if $x = \pm y$ and $y = \pm z$, then $x = \pm z$.

c) It's reflexive because $x - x = 0 \in Q$. It's symmetric because if $x - y = t \in Q$, then $y - x = -t \in Q$. It's not antisymmetric because $(3,5) \in R, (5,3) \in R$, but $3 \neq 5$. It's transitive because if $x - y = \frac{a}{b} \in Q$ and $y - z = \frac{c}{d} \in Q$, then $x - z = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \in Q$.

d) It's not reflexive because if $x = 1 \rightarrow 1 \neq 2$. It's not symmetric because if $x = 2y \rightarrow y \neq 2x$. It's antisymmetric because if $x = 2y$ and $y = 2x$, then $x = 4x \rightarrow x = y = 0$.

h) It's not reflexive because $(2,2) \notin R$. It's symmetric because if $(x,y) \in R \rightarrow x = 1 \vee y = 1 \rightarrow y = 1 \vee x = 1 \rightarrow (y,x) \in R$. It's not antisymmetric because $(1,4) \in R$ and $(4,1) \in R$, but $4 \neq 1$. It's not transitive because $(4,1) \in R$ and $(1,5) \in R$, but $(4,5) \notin R$.

5. Exercise 8 page 620

a) ISBN (or title)

b) There are no two books with the same title and the same publication date.

c) There are no two books with the same title and the same number of pages.

6. Exercise 8 page 626

a) It's not reflexive because $(2,2) \notin M_a$. It's not irreflexive because $(1,1) \in M_a$. It's symmetric. It's not antisymmetric because $(1,2) \in M_a$ and $(2,1) \in M_a$. It's not transitive because $(1,4) \in M_a$ and $(4,3) \in M_a$, but $(1,3) \notin M_a$.

b) It's reflexive. It's not irreflexive. It's not symmetric because $(1,2) \in M_b$ but $(2,1) \notin M_b$. It's antisymmetric. It's not transitive because $(1,3) \in M_b$ and $(3,4) \in M_b$, but $(1,4) \notin M_b$.

c) It's not reflexive. It's irreflexive. It's symmetric. It's not antisymmetric because $(1,2) \in M_c$ and $(2,1) \in M_c$. It's not transitive because $(1,2) \in M_c$ and $(2,3) \in M_c$, but $(1,3) \notin M_c$.

7. Exercise 26 page 638

a) $M_a = (0,0,1,0,0; 0,0,0,1,0; 1,0,0,0,0; 0,1,0,0,0; 0,0,0,1,0) \to M_a^{[5]} = (1,0,1,0,0; 0,1,0,1,0; 1,0,1,0,0; 0,1,0,1,0; 0,1,0,1,0) = \{(a,a),(a,c),(b,b),(b,d),(c,a),(c,c),(d,b),(d,d),(e,b),(e,d)\}$

b) $M_b = (0,0,0,0,0; 0,0,1,0,1; 0,0,0,0,1; 1,0,0,0,0; 0,1,1,0,0) \to M_b^{[5]} = (0,0,0,0,0; 0,1,1,0,1; 0,1,1,0,1; 1,0,0,0,0; 0,1,1,0,1) = \{(b,b),(b,c),(b,e),(c,b),(c,c),(c,e),(d,a),(e,b),(e,c),(e,e)\}$

c) $M_c = (0,1,1,0,1; 1,0,1,0,0; 1,1,0,0,0; 1,0,0,0,0; 0,0,0,1,0) \to M_c^{[5]} = (1,1,1,1,1; 1,1,1,1,1; 1,1,1,1,1; 1,1,1,1,1; 1,1,1,1,1) = \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,a),(b,b),(b,c),(b,d),(b,e),$
$(c,a),(c,b),(c,c),(c,d),(c,e),(d,a),(d,b),(d,c),(d,d),(d,e),$
$(e,a),(e,b),(e,c),(e,d),(e,e)\}$

d) $M_d = (0,0,0,0,1; 1,0,0,1,0; 0,0,0,1,0; 1,0,1,0,0; 1,1,1,0,1) \to M_d^{[5]} = (1,1,1,1,1; 1,1,1,1,1; 1,1,1,1,1; 1,1,1,1,1; 1,1,1,1,1) = \{(a,a),(a,b),(a,c),(a,d),(a,e),(b,a),(b,b),(b,c),(b,d),(b,e),$
$(c,a),(c,b),(c,c),(c,d),(c,e),(d,a),(d,b),(d,c),(d,d),(d,e),$
$(e,a),(e,b),(e,c),(e,d),(e,e)\}$

8. Exercise 16 page 647

$ab = ba \to \big((a,b),(a,b)\big) \in R \to$ reflexive

$\big((a,b),(c,d)\big) \in R \to ad = bc \to cb = da \to \big((c,d),(a,b)\big) \in R \to$ symmetric

$$\begin{cases} ((a,b),(c,d)) \in R \rightarrow ad = bc \rightarrow \dfrac{d}{c} = \dfrac{b}{a} \\ ((c,d),(e,f)) \in R \rightarrow cf = de \rightarrow \dfrac{d}{c} = \dfrac{f}{e} \end{cases}$$

$\rightarrow \frac{b}{a} = \frac{f}{e} \rightarrow af = be \rightarrow ((a,b),(e,f)) \in R \rightarrow$ transitive

So R is an equivalence relation.

9. Exercise 6 page 662

a) It's a poset because it's reflexive, antisymmetric and transitive.

b) It's not a poset because it's not reflexive.

c) It's a poset because it's reflexive, antisymmetric and transitive.

d) It's not a poset because it's not reflexive.

10. Exercise 10 page 683

3) It's simple.
4) Remove one of the edges between (a,b). And remove two edges between (b,d).
5) Remove loops and one edge from (a,b) and one from (b,d) and one from (c,d).
6) Remove one edge from (a,c) and one edge from (b,d).
7) Make all the edges undirected and remove loops. Remove one edge from (c,d).
8) Make all the edges undirected and remove loops. Remove one edge from (a,b), one from (a,e), one from (b,c), and two from (c,d).
9) Make all the edges undirected and remove loops. Remove one edge from (a,b), one from (e,d), and one from (b,c).

11. Exercise 36 page 701

The subgraph induced by a subset W of the vertex set V is the graph (W,F), where the edge set F contains an edge in E if and only if both endpoints of this edge are in W. So a subgraph induced by a nonempty set of K_n has all the edges between the selected vertices from K_n and therefore it's a complete graph.