CS220/MATH320 – Applied Discrete Mathematics

Instructor: Ramin Dehghanpoor

Practice Exam

Name: ____

Question 1: Cardinality

How many distinct elements does the set S contain in each case? Check the appropriate box.

set description	number of elements					
	0	1	2	3	4	≥5
a) $S = \{7, 2, 3\} \cup \{3, 1, 2\}$	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]
b) $S = \{(x, y), (y, z), (z, z)\} \cap \{(y, x), (z, z), (y, y)\}$	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]
c) $S = \{A \mid (A \subseteq \{1, 2, 3, 4\}) \wedge (\lvert A \rvert = 5)\}$	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]
d) $S = \{x \mid x^2 + 2x = 8;\ x \text{ is a real number}\}$	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]
e) $S = \{(a, b) \mid a < b;\ a, b \in \{1, 2, 3\}\}$	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]
f) $S = E$, where $G = (V, E)$ is a tree and $\lvert V \rvert = 5$	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]
g) $S = \{G \mid G \text{ is a simple graph with 4 vertices}\}$	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]
h) $S = \{R \mid R \text{ is a reflexive relation on } \{0, 1\}\}$	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]
i) $S = \{n \mid (n \text{ is prime}) \wedge (n \bmod 2 = 0)\}$	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]
j) $S = \{\text{a, b, c, e}\} - \{\text{b, c, d}\}$	[ ]	[ ]	[ ]	[ ]	[ ]	[ ]

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Question 2: Equivalence Relations

Are the following relations on the set of all people equivalence relations? If not, give a reason.

a) R = {(a, b) | a and b are the same age}

b) R = {(a, b) | a and b have the same parents}

c) R = {(a, b) | a and b share a common parent}

d) R = {(a, b) | a and b have met}

e) R = {(a, b) | a and b speak a common language}

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Question 3: About Urns and Balls

a) There is an urn containing four blue balls and four red balls. We randomly draw four balls from this urn without returning any balls. What is the probability that all of the four balls that we drew are blue?

b) There are two urns, each of them containing two blue balls and two red balls. We randomly draw two balls from the first urn and then randomly draw two balls from the second urn, without returning any balls. What is the probability that all of the four balls that we drew are blue?

c) There are four urns, each of them containing one blue ball and one red ball. We randomly draw one ball from each urn without returning any balls. What is the probability that all of the four balls that we drew are blue?

d) In each of the previous three experiments, you calculated the probability of randomly drawing exactly four blue balls from a set of four blue and four red balls. Explain in English words why this probability changes with the number of urns as you observed it. Why does the probability increase(?)/decrease(?) with a larger number of urns instead of being constant?

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Question 4: Rapid Reproduction

Somewhere in the forests of Massachusetts, scientists from UMass Boston discovered two rare species of animals named the Violent Squirrel and the Sarcastic Cat. On their first encounter with these animals, the scientists found five animals of each species. One year later, the scientists returned and then found five Violent Squirrels and 13 Sarcastic Cats. The scientists somehow devised formulas for the populations v_n and s_n, denoting the number of Violent Squirrels and Sarcastic Cats, respectively, in year n, for $n \geq 2$:

$v_n = n \cdot v_{n-1}$

$s_n = 4s_{n-1} + 5s_{n-2}$

a) Let us define that the species were discovered in year 0, and the second counting was done in year 1. Use the above formulas to predict the populations v_n and s_n in the years n = 2, 3, 4, and 5.

b) Find explicit formulas for v_n and s_n, $n \geq 2$, that do not require iteration. You should (but do not have to) check the correctness of your formulas using some of the results you obtained in a).

c) Describe the growth of v_n and s_n using the big-O notation for each of them. In each estimate O(f(n)), f(n) should be the most suitable function chosen from the following ones: $\log n$, n, $n \log n$, n^2, n^3, 2^n, 3^n, 4^n, 5^n, 6^n, $n!$, n^n.

d) In the year 1,000, will there be more Violent Squirrels than Sarcastic Cats, given that the populations develop as predicted? Or will there be more Sarcastic Cats than Violent Squirrels? Do not try to compute the actual numbers! Just tell which species you think will have the larger population, and give the reason why you think so.

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Question 5: Possibilities

a) Paul wants to come to the U.S. to first get his Bachelor's degree, then his Master's degree, and finally his Ph.D. He considers seven possible universities for his studies, but he does not want to receive more than one degree from the same university. How many different academic career paths are possible for Paul, assuming that he does not switch universities during each program and actually passes all three programs?

b) Ten years later, Paul becomes a professor at UMass Boston. He receives seven applications from students who want to work in his new Time Travel Lab, but his research grants are limited, and so he will hire only four of them. How many choices in the recruitment of his research team does Paul have?

c) Paul's most famous research article is about four-letter strings made up of the letters 'a', 'b', and 'c'. He discovers the number of such strings in which an 'a' is never immediately followed by another 'a' or a 'b', and a 'b' or 'c' is never immediately followed by a 'c'. Use a tree diagram to replicate Paul's finding. How many different strings of this kind are there?

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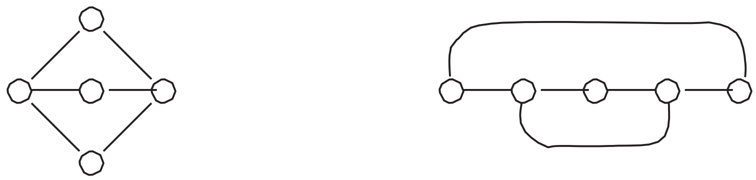
Question 6: Lack of Isomorphism

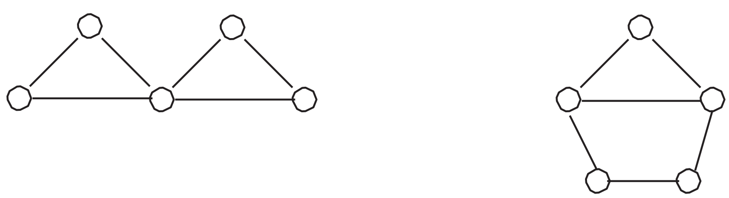
The following pairs of simple graphs are non-isomorphic. For each pair, give a precise reason why it is impossible that the two graphs are isomorphic.

a)



b)

c)



d)

e)

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Question 7: About Trees

(a) If T is a binary tree with 41 vertices, its minimum height is ______ .

(b) If T is a full binary tree with 111 vertices, its maximum height is ______ .

(c) Every full binary tree with 51 vertices has _______ leaves.

(d) Every full binary tree with 60 leaves has _______ vertices.

(e) Every full binary tree with 75 vertices has _______ internal vertices.

(f) A full 3-ary tree with 100 internal vertices has _______ vertices.

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Question 8 (Bonus): Independence Day

(a) Give an example for two independent events. Prove their independence mathematically.

(b) Give an example for two events that are not independent. Prove mathematically that they are not independent.