CS220/MATH320 – Applied Discrete Mathematics

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Practice Exam

Sample Solutions

Name: ____________________________________

Question 1: Cardinality

How many distinct elements does the set S contain in each case? Check the appropriate box.

set description	number of elements					
	0	1	2	3	4	≥5
a) $S = \{7, 2, 3\} \cup \{3, 1, 2\}$	[ ]	[ ]	[ ]	[ ]	[X]	[ ]
b) $S = \{(x, y), (y, z), (z, z)\} \cap \{(y, x), (z, z), (y, y)\}$	[ ]	[X]	[ ]	[ ]	[ ]	[ ]
c) $S = \{A \mid (A \subseteq \{1, 2, 3, 4\}) \wedge (\|A\| = 5)\}$	[X]	[ ]	[ ]	[ ]	[ ]	[ ]
d) $S = \{x \mid x^2 + 2x = 8;\ x \text{ is a real number}\}$	[ ]	[ ]	[X]	[ ]	[ ]	[ ]
e) $S = \{(a, b) \mid a < b;\ a, b \in \{1, 2, 3\}\}$	[ ]	[ ]	[ ]	[X]	[ ]	[ ]
f) $S = E$, where $G = (V, E)$ is a tree and $\|V\| = 5$	[ ]	[ ]	[ ]	[ ]	[X]	[ ]
g) $S = \{G \mid G \text{ is a simple graph with 4 vertices}\}$	[ ]	[ ]	[ ]	[ ]	[ ]	[X]
h) $S = \{R \mid R \text{ is a reflexive relation on } \{0, 1\}\}$	[ ]	[ ]	[ ]	[ ]	[X]	[ ]
i) $S = \{n \mid (n \text{ is prime}) \wedge (n \bmod 2 = 0)\}$	[ ]	[X]	[ ]	[ ]	[ ]	[ ]
j) $S = \{\text{a, b, c, e}\} - \{\text{b, c, d}\}$	[ ]	[ ]	[X]	[ ]	[ ]	[ ]

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Question 2: Equivalence Relations

Are the following relations on the set of all people equivalence relations? If not, give a reason.

a) R = {(a, b) | a and b are the same age}

Yes.

b) R = {(a, b) | a and b have the same parents}

Yes.

c) R = {(a, b) | a and b share a common parent}

No, it is not transitive. a and b could have the same father, and b and c could have the same mother, without a and c sharing a common parent.

d) R = {(a, b) | a and b have met}

No, it is not transitive. If a and b have met, and b and c have met, it does not necessarily mean that a and c have met. You could also argue that it is not reflexive, because nobody meets herself or himself.

e) R = {(a, b) | a and b speak a common language}

No, it is not transitive. If a and b both speak German, and b and c both speak Klingon, it does not imply that a and c speak a common language.

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Question 3: About Urns and Balls

a) There is an urn containing four blue balls and four red balls. We randomly draw four balls from this urn without returning any balls. What is the probability that all of the four balls that we drew are blue?

For the first ball to be blue, the probability is 4/8. Once the first blue ball has been removed, the chance for the next one to be blue is 3/7. For the remaining two balls it is 2/6 and 1/5. Since all of these things need to happen, we apply the product rule:

$p = 4/8 \cdot 3/7 \cdot 2/6 \cdot 1/5 = 1/70$

b) There are two urns, each of them containing two blue balls and two red balls. We randomly draw two balls from the first urn and then randomly draw two balls from the second urn, without returning any balls. What is the probability that all of the four balls that we drew are blue?

For the first blue ball, the chance is 2/4, and for the second it is 1/3. The same applies to the second urn. So we get:

$p = 2/4 \cdot 1/3 \cdot 2/4 \cdot 1/3 = 1/36$

c) There are four urns, each of them containing one blue ball and one red ball. We randomly draw one ball from each urn without returning any balls. What is the probability that all of the four balls that we drew are blue?

In each of the four cases, the probability of picking a blue ball is ½. Therefore:

$p = 1/2 \cdot 1/2 \cdot 1/2 \cdot 1/2 = 1/16$

d) In each of the previous three experiments, you calculated the probability of randomly drawing exactly four blue balls from a set of four blue and four red balls. Explain in English words why this probability changes with the number of urns as you observed it. Why does the probability increase(?)/decrease(?) with a larger number of urns instead of being constant?

The probability of picking four blue balls increases with the number of urns they are divided among, because whenever we are done with one urn, the red balls in it cannot be picked anymore. For example, if we have a single urn, the four red balls make it less and less likely for us to pick a blue ball as the blue balls are being removed. If, on the other hand, we have four urns, each time we pick a blue ball, the red ball that shared an urn with it can now longer be picked.

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Question 4: Rapid Reproduction

Somewhere in the forests of Massachusetts, scientists from UMass Boston discovered two rare species of animals named the Violent Squirrel and the Sarcastic Cat. On their first encounter with these animals, the scientists found five animals of each species. One year later, the scientists returned and then found five Violent Squirrels and 13 Sarcastic Cats. The scientists somehow devised formulas for the populations v_n and s_n, denoting the number of Violent Squirrels and Sarcastic Cats, respectively, in year n, for $n \geq 2$:

$v_n = n \cdot v_{n-1}$

$s_n = 4s_{n-1} + 5s_{n-2}$

a) Let us define that the species were discovered in year 0, and the second counting was done in year 1. Use the above formulas to predict the populations v_n and s_n in the years $n = 2, 3, 4$, and 5.

$v_2 = 2v_1 = 2 \cdot 5 = 10$

$v_3 = 3v_2 = 3 \cdot 10 = 30$

$v_4 = 4v_3 = 4 \cdot 30 = 120$

$v_5 = 5v_4 = 5 \cdot 120 = 600$

$s_2 = 4s_1 + 5s_0 = 4 \cdot 13 + 5 \cdot 5 = 52 + 25 = 77$

$s_3 = 4s_2 + 5s_1 = 4 \cdot 77 + 5 \cdot 13 = 308 + 65 = 373$

$s_4 = 4s_3 + 5s_2 = 4 \cdot 373 + 5 \cdot 77 = 1492 + 385 = 1877$

$s_5 = 4s_4 + 5s_3 = 4 \cdot 1877 + 5 \cdot 373 = 7508 + 1865 = 9373$

b) Find explicit formulas for v_n and s_n, $n \geq 2$, that do not require iteration. You should (but do not have to) check the correctness of your formulas using some of the results you obtained in a).

Characteristic equation for sarcastic cats: $r^2 - 4r - 5 = 0$

$(r - 5)(r + 1) = 0$

$r_1 = 5, r_2 = (-1)$

Solution:

$s_n = \alpha_1 \cdot 5^n + \alpha_2 \cdot (-1)^n$

Given the initial conditions:

$s_0 = \alpha_1 + \alpha_2 = 5$

$s_1 = 5\alpha_1 - \alpha_2 = 13$

$\Rightarrow \alpha_1 = 3, \alpha_2 = 2$

Result:

$s_n = 3 \cdot 5^n + 2 \cdot (-1)^n$

For the violent squirrels:

$v_n = n \cdot v_{n-1}$

Given the initial conditions:

$v_n = 5n!$

c) Describe the growth of v_n and s_n using the big-O notation for each of them. In each estimate O(f(n)), f(n) should be the most suitable function chosen from the following ones: log n, n, n log n, n^2, n^3, 2^n, 3^n, 4^n, 5^n, 6^n, $n!$, n^n.

$s_n = O(5^n)$

$v_n = O(n!)$

d) In the year 1,000, will there be more Violent Squirrels than Sarcastic Cats, given that the populations develop as predicted? Or will there be more Sarcastic Cats than Violent Squirrels? Do not try to compute the actual numbers! Just tell which species you think will have the larger population, and give the reason why you think so.

There will be more violent squirrels than sarcastic cats, because O(n!) indicates faster growth than does $O(5^n)$.

Question 5: Possibilities

a) Paul wants to come to the U.S. to first get his Bachelor's degree, then his Master's degree, and finally his Ph.D. He considers seven possible universities for his studies, but he does not want to receive more than one degree from the same university. How many different academic career paths are possible for Paul, assuming that he does not switch universities during each program and actually passes all three programs?

Here, Paul has 7 choices for his Bachelor's degree, which leaves him with 6 choices for his Master's and 5 choices for his Ph.D. So the total number of career paths is:

$7 \cdot 6 \cdot 5 = 210$

b) Ten years later, Paul becomes a professor at UMass Boston. He receives seven applications from students who want to work in his new Time Travel Lab, but his research grants are limited, and so he will hire only four of them. How many choices in the recruitment of his research team does Paul have?

The order of recruitment does not matter here, so the answer is:
$C(7, 4) = 7!/(4! \cdot 3!) = 35$ choices

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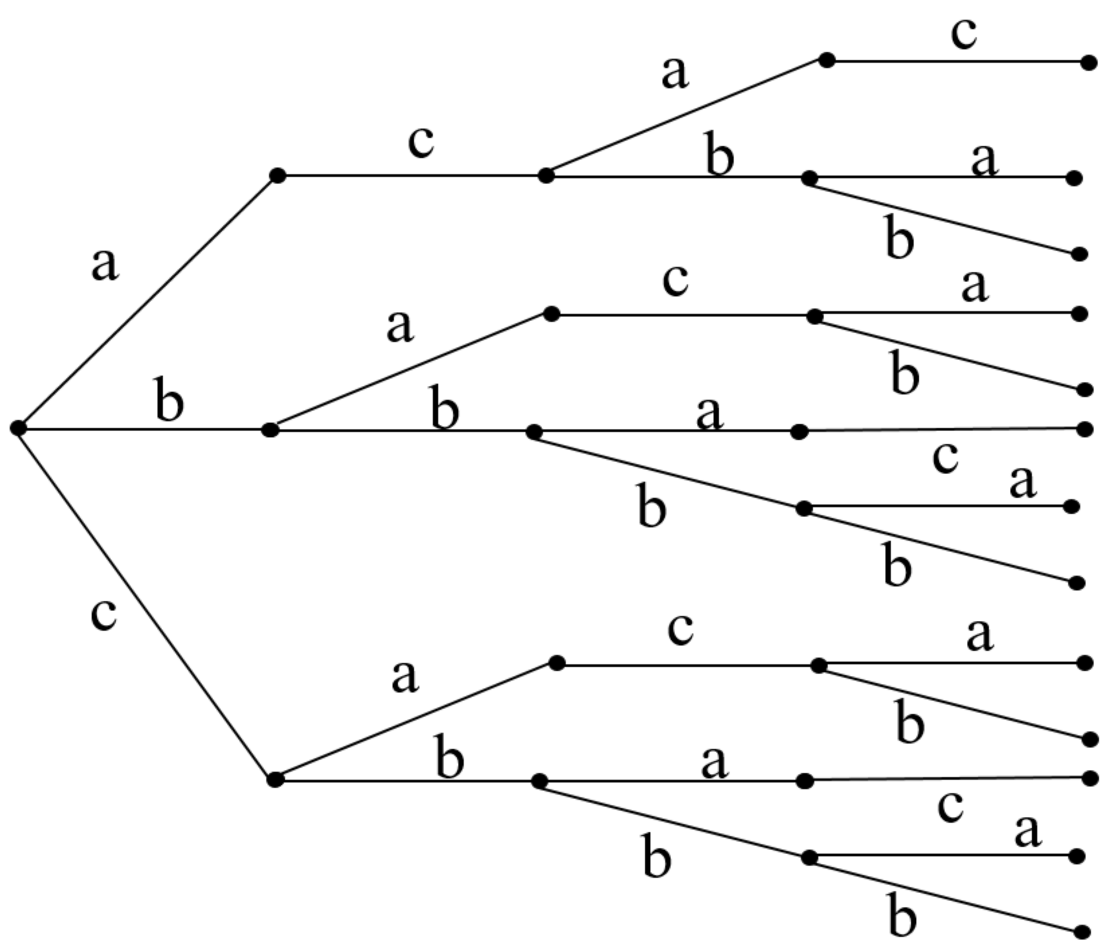
c) Paul's most famous research article is about four-letter strings made up of the letters 'a', 'b', and 'c'. He discovers the number of such strings in which an 'a' is never immediately followed by another 'a' or a 'b', and a 'b' or 'c' is never immediately followed by a 'c'. Use a tree diagram to replicate Paul's finding. How many different strings of this kind are there?

There are 13 strings:



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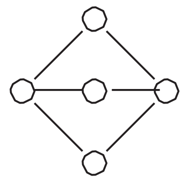
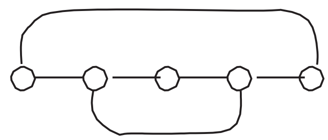
Question 6: Lack of Isomorphism

The following pairs of simple graphs are non-isomorphic. For each pair, give a precise reason why it is impossible that the two graphs are isomorphic.

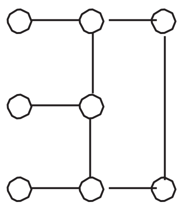
a)

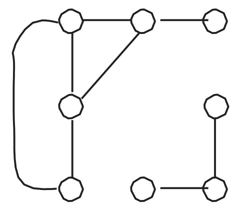




Right graph has a circuit of length 3, left graph does not.

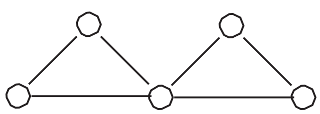
b)

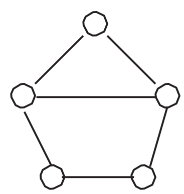




Left graph is connected, right graph is not.

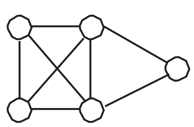
c)

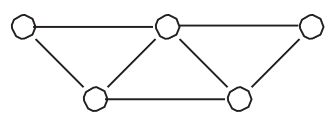




Left graph has a vertex of degree 4, right graph does not.

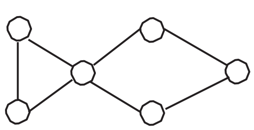
d)

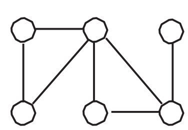




Left graph has eight edges, right graph has only seven.

e)





Right graph has a vertex of degree one, left graph does not.

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Question 7: About Trees

(a) If T is a binary tree with 41 vertices, its minimum height is 5.

(b) If T is a full binary tree with 111 vertices, its maximum height is 55.

(c) Every full binary tree with 51 vertices has 26 leaves.

(d) Every full binary tree with 60 leaves has 119 vertices.

(e) Every full binary tree with 75 vertices has 37 internal vertices.

(f) A full 3-ary tree with 100 internal vertices has 301 vertices.

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Question 8 (Bonus): Independence Day

(a) Give an example for two independent events. Prove their independence mathematically.

Rolling two fair dice A and B. The event E that die A shows a 6 is independent of the event F that die B shows a 6.

Proof:

$p(E) = 1/6$, $p(F) = 1/6$, $p(E \cap F) = 1/36$, so we have:

$p(E) \cdot p(F) = p(E \cap F)$, so the events are independent.

(b) Give an example for two events that are not independent. Prove mathematically that they are not independent.

Rolling a single die twice. The event G that the first number I roll is a 6 is not independent from the event H that the sum of the two numbers I roll is 12.

Proof:

$p(G) = 1/6$, $p(H) = 1/36$, $p(G \cap H) = 1/36$, so we have:

$p(G) \cdot p(H) \neq p(G \cap H)$, so the events are not independent.