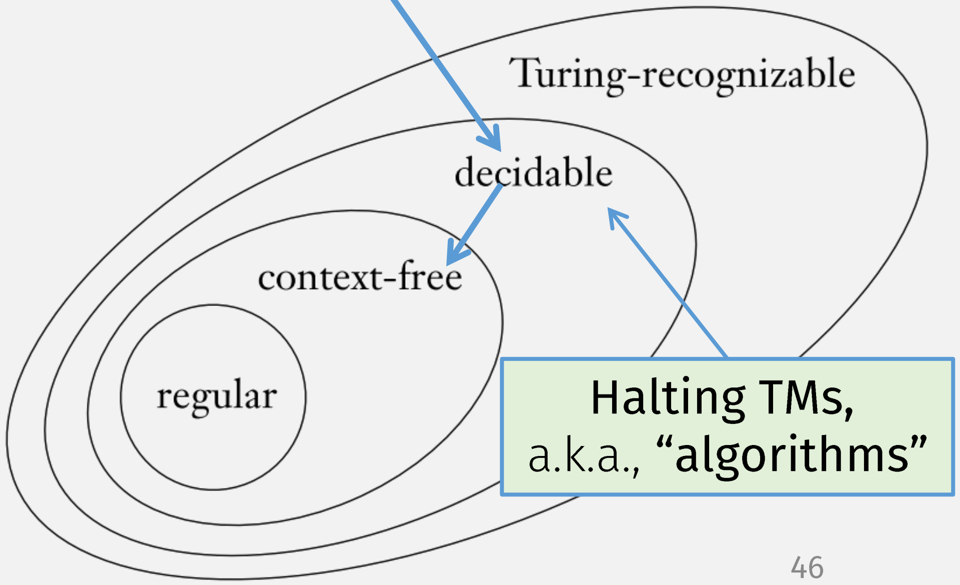
UMB CS 420

Decidability for CFLs

November 8, 2022



Announcements

- HW 7 in
  - ~~Due Mon Nov 7 11:59pm EST~~

- HW 8 out
  - Due Mon Nov 14 11:59pm EST

Last Time: Decider Turing Machines

- 2 classes of Turing Machines
  - **Recognizers** (all TMs): may loop forever
    - A TM that loops on an input does not accept that input
  - **Deciders** (subset of TMs) (**algorithms**) always halt
    - And then either **accept** or **reject**

- **Decider** definitions must include a **termination argument:**
  - Explains (informally) why every step in the TM halts
  - (Pay special attention to loops)

Last Time: Decidable Languages About DFAs

- $A_{\mathsf{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string}$
  - Decider TM: implements B DFA's extended δ fn

Remember:
TMs = programs
Creating TM = programming
Previous theorems = library

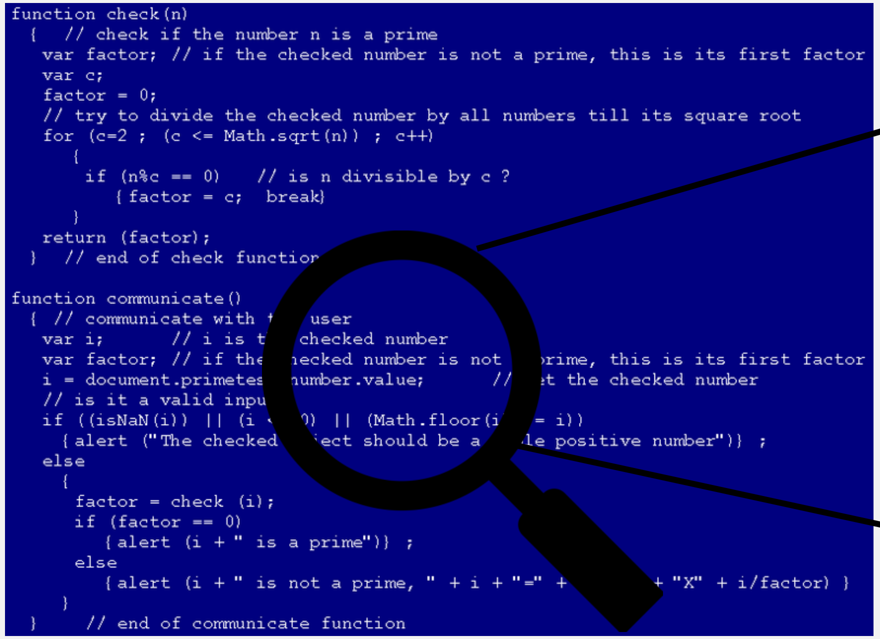
- $A_{\mathsf{NFA}} = \{\langle B, w\rangle \mid B \text{ is an NFA that accepts input string } w\}$
  - Decider TM: uses **NFA→DFA** algorithm + A_{DFA} decider

- $A_{\mathsf{REX}} = \{\langle R, w\rangle \mid R \text{ is a regular expression that generates string } w\}$
  - Decider TM: uses **RegExpr→NFA** algorithm + A_{NFA} decider

Flashback: Why Study Algorithms About Computing

2. **To predict what programs will do**
   (without running them!)

Not possible for all programs! But …



???

Predicting What <u>Some</u> Programs Will Do …

What if: look at <u>weaker</u> computation models
… like DFAs and regular languages!

Thm: E_{DFA} is a decidable language

$$E_{\text{DFA}} = \{\langle A \rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$$

E_{DFA} is a language … of DFA descriptions,
i.e., $(Q, \Sigma, \delta, q_0, F)$ …

… where the language of each DFA …
must be { }, i.e., DFA accepts no strings

Is there a decider that
accepts/rejects DFA descriptions …

… by predicting something
about the DFA's language
(by analyzing its description)

The key question we are studying:
Can we compute something about the computation of a program, by analyzing only its source code?

Analogy
DFA's description : a program's source code ::
DFA's language : what the program computes

Important: **don't confuse the different languages here!**

Thm: E_{DFA} is a decidable language

$E_{\mathsf{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$

Decider:

T = "On input $\langle A\rangle$, where A is a DFA:

1. Mark the start state of A.
2. Repeat until no new states get marked:
3. Mark any state that has a transition coming into it from any state that is already marked.
4. If no accept state is marked, *accept*; otherwise, *reject*."

If loop marks at least 1 state on each iteration, then it eventually terminates because there are finite states; else loop terminates

Termination argument?

I.e., this is a "reachability" algorithm …

… check if accept states are "reachable" from start state

Note: Machine does not "run" the DFA!

… it computes something about the DFA's language (computation) by analyzing it's description (source code)

Thm: EQ_{DFA} is a decidable language

$$EQ_{\text{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$$

I.e., Can we compute whether two DFAs are "equivalent"?

Replacing "**DFA**" with "**program**" =
A "**holy grail**" **of computer science**!

Thm: EQ_{DFA} is a decidable language

$EQ_{\text{DFA}} = \{\langle A, B\rangle |\ A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$

A Naïve Attempt (assume alphabet **{a}**):

1. Simulate:
   - A **with input a,** and
   - B **with input a**
   - **Reject** if results are different, else …
2. Simulate :
   - A **with input aa,** and
   - B **with input aa**
   - **Reject** if results are different, else …

- …

This might not terminate!
(Hence it's not a decider)

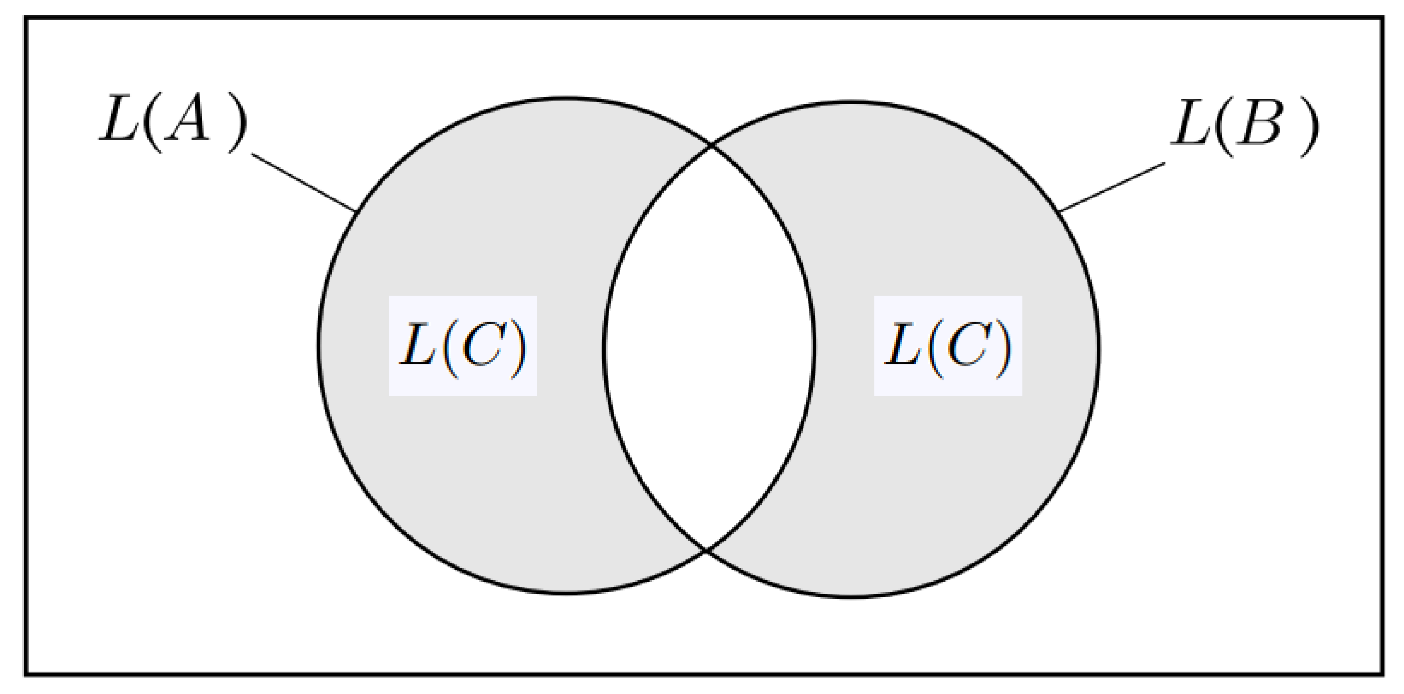
Can we compute this without simulating running the DFAs?

Thm: EQ_{DFA} is a decidable language

$$EQ_{\mathsf{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$$

Trick: Use Symmetric Difference

Symmetric Difference



$$L(C) = \left(L(A) \cap \overline{L(B)}\right) \cup \left(\overline{L(A)} \cap L(B)\right)$$

$$\boxed{L(C) = \emptyset \text{ iff } L(A) = L(B)}$$

Thm: EQ_{DFA} is a decidable language

$EQ_{\text{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$

NOTE: This only works because: **negation**, i.e., set complement, and **intersection** is **closed** for regular languages

Construct **decider** using 2 parts:

1. Symmetric Difference algo: $L(C) = \left(L(A) \cap \overline{L(B)}\right) \cup \left(\overline{L(A)} \cap L(B)\right)$
   - Construct C = Union, intersection, negation of machines A and B
2. Decider T (from "library") for: $E_{\text{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$
   - Because $L(C) = \emptyset$ iff $L(A) = L(B)$

F = "On input $\langle A, B\rangle$, where A and B are DFAs:

1. Construct DFA C as described.
2. Run TM T deciding E_{DFA} on input $\langle C\rangle$.
3. If T accepts, *accept*. If T rejects, *reject*."

Termination argument?

Predicting What Some Programs Will Do …

microsoft.com/en-us/research/project/slam/

SLAM is a project for checking that software satisfies critical behavioral properties of the interfaces it uses and to aid software engineers in designing interfaces and software that ensure reliable and correct functioning. Static Driver Verifier is a tool in the Windows Driver Development Kit that uses the SLAM verification engine.

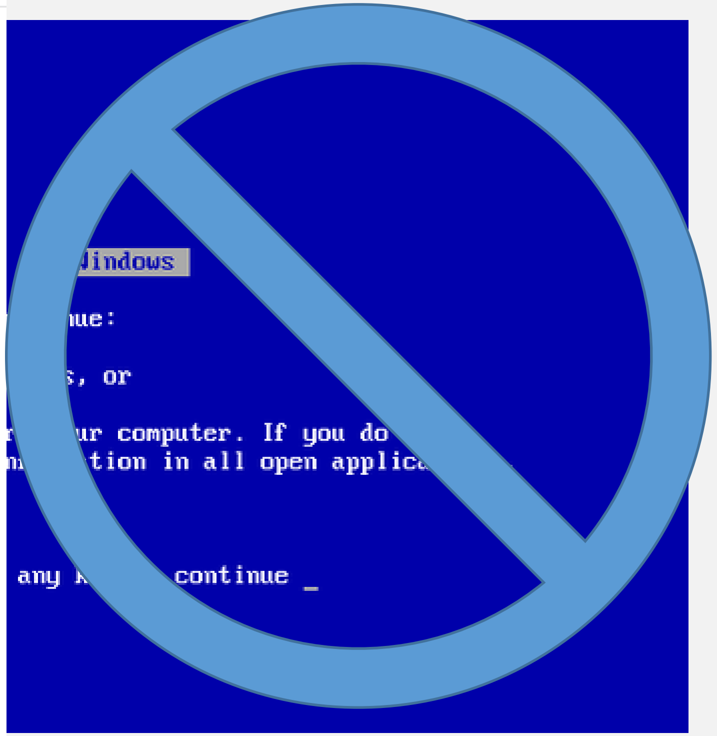
"Things like even software verification, this has been the Holy Grail of computer science for many decades but now in some very key areas, for example, driver verification we're building tools that can do actual proof about the software and how it works in order to guarantee the reliability." **Bill Gates, April 18, 2002. Keynote address at WinHec 2002**

Static Driver Verifier Research Platform README

Overview of Static Driver Verifier Research Platform

Static Driver Verifier (SDV) is a compile-time static verification
Research Platform (SDVRP) is an extension to SDV that allows

- Support additional frameworks (or APIs) and write custo
- Experiment with the model checking step.



Model checking

From Wikipedia, the free encyclopedia

In computer science, **model checking** or **property checking** is a method for checking whether a finite-state model of a system meets a given specification (also known as correctness). This is typically

Its "language"

Summary: Algorithms About Regular Langs

- $A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$
  - **Decider**: Simulates DFA by implementing extended δ function
- $A_{\text{NFA}} = \{\langle B, w\rangle \mid B \text{ is an NFA that accepts input string } w\}$
  - **Decider**: Uses **NFA→DFA** decider + A_{DFA} decider
- $A_{\text{REX}} = \{\langle R, w\rangle \mid R \text{ is a regular expression that generates string } w\}$
  - **Decider**: Uses **RegExpr→NFA** decider + A_{NFA} decider
- $E_{\text{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$
  - **Decider**: Reachability algorithm

Lang of the DFA

- $EQ_{\text{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$

  - **Decider**: Uses complement and intersection closure construction + E_{DFA} decider

Remember:
TMs = programs
Creating TM = programming
Previous theorems = library

Next: Algorithms (Decider TMs) for CFLs?

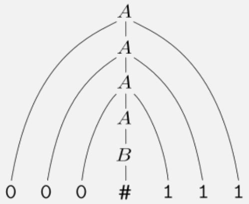
- What can we predict about CFGs or PDAs?

Thm: A_{CFG} is a decidable language

$A_{\mathsf{CFG}} = \{\langle G, w\rangle |\ G \text{ is a CFG that generates string } w\}$

- This is a very practically important problem …
- … equivalent to:
  - **Algorithm** to **parse** "program" w for PL with grammar G?

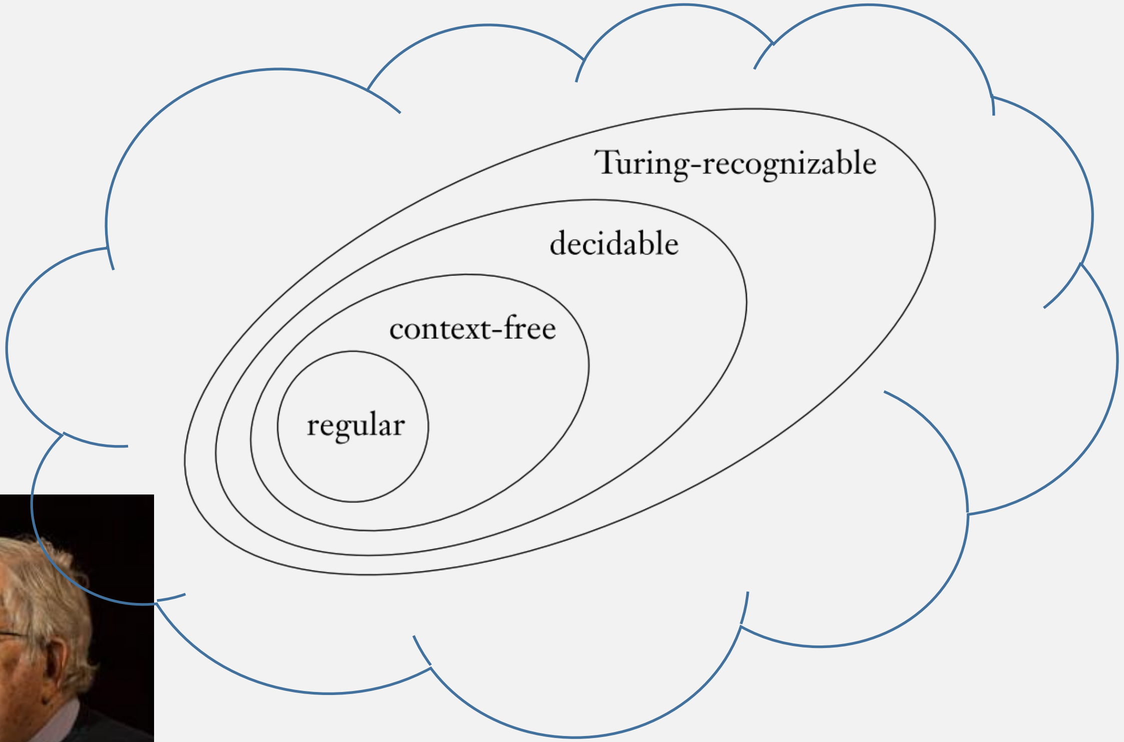
- A Decider for this problem could … ?
  - Try every possible derivation of G, and check if it's equal to w?
  - But this might never halt
    - E.g., what if there are rules like: $\mathbf{S} \to \mathbf{0S}$ or $\mathbf{S} \to \mathbf{S}$
  - This TM would be a recognizer but not a decider



Idea: can the TM stop checking after some length?
- I.e., Is there upper bound on the number of derivation steps?

Chomsky Normal Form

Noam Chomsky



He (sort of) invented this course too!

Chomsky Normal Form

A context-free grammar is in ***Chomsky normal form*** if every rule is of the form

$$A \to BC$$
$$A \to a$$

(non-start) Variables only

2 rule shapes

Terminals only

where a is any terminal and A, B, and C are any variables—except that B and C may not be the start variable. In addition, we permit the rule $S \to \varepsilon$, where S is the start variable.

Chomsky Normal Form Example

Makes the string long enough

Convert variables to terminals

- $S \to AS \mid AB$
- $A \to \mathtt{a}$
- $B \to \mathtt{b}$

- To generate string of length: 2
  - Use S rule: 1 time; Use A or B rules: 2 times
  - $S \Rightarrow AB \Rightarrow \mathtt{a}B \Rightarrow \mathtt{ab}$
  - Derivation total steps: $1 + 2 = \boxed{3}$
- To generate string of length: 3
  - Use S rule: 2 times; A or B rules: 3 times
  - $S \Rightarrow AS \Rightarrow AAB \Rightarrow \mathtt{a}AB \Rightarrow \mathtt{aa}B \Rightarrow \mathtt{aab}$
  - Derivation total steps: $2 + 3 = \boxed{5}$
- To generate string of length: 4
  - Use S rule: 3 times; A or B rules: 4 times
  - $S \Rightarrow AS \Rightarrow AAS \Rightarrow AAAB \Rightarrow \mathtt{a}AAB \Rightarrow \mathtt{aa}AB \Rightarrow \mathtt{aaa}B \Rightarrow \mathtt{aaab}$
  - Derivation total steps: $3 + 4 = \boxed{7}$
- …

A context-free grammar is in ***Chomsky normal form*** if every rule is of the form

$$A \to BC$$
$$A \to a$$

2 rule shapes

where a is any terminal and A, B, and C are any variables—except that B and C may not be the start variable. In addition, we permit the rule $S \to \varepsilon$, where S is the start variable.

Chomsky Normal Form: Number of Steps

To generate a string of length n:

$n - 1$ steps: to generate n variables

Makes the string long enough

$+\ n$ steps: to turn each variable into a terminal

Convert string to terminals

Total: $2n - 1$ steps

(A *finite* number of steps!)

Chomsky normal form

$A \to BC$ Use n-1 times

$A \to a$ Use n times

Thm: A_{CFG} is a decidable language

$A_{\mathsf{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}$

Proof: create the decider:

$S =$ "On input $\langle G, w\rangle$, where G is a CFG and w is a string:

1. Convert G to an equivalent grammar in Chomsky normal form.
2. List all derivations with $\boxed{2n - 1 \text{ steps,}}$ where n is the length of w; except if $n = 0$, then instead list all derivations with one step.
3. If any of these derivations generate w, *accept*; if not, *reject*."

We first need to prove this is true for all CFGs!

Step 1: Conversion to Chomsky Normal Form is an algorithm …
Step 2:
Step 3:

Termination argument?

Thm: Every CFG has a Chomsky Normal Form

Proof: Create algorithm to convert any CFG into Chomsky Normal Form

Chomsky normal form

$A \to BC$
$A \to a$

1. Add new start variable S_0 that does not appear on any RHS
   - I.e., add rule $S_0 \to S$, where S is old start var

$$\begin{aligned} S &\to ASA \mid \mathtt{a}B \\ A &\to B \mid S \\ B &\to \mathtt{b} \mid \varepsilon \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} &\boxed{\boldsymbol{S_0 \to S}} \\ S &\to ASA \mid \mathtt{a}B \\ A &\to B \mid S \\ B &\to \mathtt{b} \mid \varepsilon \end{aligned}$$

Thm: Every CFG has a Chomsky Normal Form

Chomsky normal form

$A \to BC$
$A \to a$

1. Add new start variable S_0 that does not appear on any RHS
   - I.e., add rule $S_0 \to S$, where S is old start var
2. Remove all "empty" rules of the form $A \to \varepsilon$
   - A must not be the start variable
   - Then for every rule with A on RHS, add new rule with A deleted
     - E.g., If $R \to uAv$ is a rule, add $R \to uv$
   - Must cover all combinations if A appears more than once in a RHS
     - E.g., if $R \to uAvAw$ is a rule, add 3 rules: $R \to uvAw$, $R \to uAvw$, $R \to uvw$

$$\begin{aligned} S_0 &\to S \\ S &\to ASA \mid \mathtt{a}B \mid \mathbf{a} \\ A &\to B \mid S \mid \boldsymbol{\varepsilon} \\ B &\to \mathtt{b} \mid \varepsilon \end{aligned}$$

Then, add

First, remove

$$\begin{aligned} S_0 &\to S \\ S &\to ASA \mid \mathtt{a}B \mid \mathtt{a} \mid \boldsymbol{SA} \mid \boldsymbol{AS} \mid \boldsymbol{S} \\ A &\to B \mid S \mid \varepsilon \\ B &\to \mathtt{b} \end{aligned}$$

Then add, to account for possibly empty A

Then, remove

Thm: Every CFG has a Chomsky Normal Form

Chomsky normal form

$A \to BC$
$A \to a$

1. Add new start variable S_0 that does not appear on any RHS
   - I.e., add rule $S_0 \rightarrow S$, where S is old start var
2. Remove all "empty" rules of the form $A \rightarrow \varepsilon$
   - A must not be the start variable
   - Then for every rule with A on RHS, add new rule with A deleted
     - E.g., If $R \rightarrow uAv$ is a rule, add $R \rightarrow uv$
   - Must cover all combinations if A appears more than once in a RHS
     - E.g., if $R \rightarrow uAvAw$ is a rule, add 3 rules: $R \rightarrow uvAw$, $R \rightarrow uAvw$, R -> uvw
3. Remove all "unit" rules of the form $A \rightarrow B$
   - Then, for every rule $B \rightarrow u$, add rule $A \rightarrow u$

$$\begin{aligned} S_0 &\to S \\ S &\to ASA \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \mid S \\ A &\to B \mid S \\ B &\to \mathtt{b} \end{aligned}$$

Remove, no add (same variable)

$$\begin{aligned} S_0 &\to S \mid \boldsymbol{ASA} \mid \mathbf{a}\boldsymbol{B} \mid \mathbf{a} \mid \boldsymbol{SA} \mid \boldsymbol{AS} \\ S &\to ASA \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \\ A &\to B \mid S \\ B &\to \mathtt{b} \end{aligned}$$

Remove, then add S RHSs to S_0

$$\begin{aligned} S_0 &\to ASA \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \\ S &\to ASA \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \\ A &\to S \mid \mathtt{b} \mid \boldsymbol{ASA} \mid \mathbf{a}\boldsymbol{B} \mid \mathbf{a} \mid \boldsymbol{SA} \mid \boldsymbol{AS} \\ B &\to \mathtt{b} \end{aligned}$$

Remove, then add S RHSs to A

Termination argument of this algorithm?

Thm: Every CFG has a Chomsky Normal Form

Chomsky normal form

$A \to BC$
$A \to a$

1. Add new start variable S_0 that does not appear on any RHS
   - I.e., add rule $S_0 \to S$, where S is old start var
2. Remove all "empty" rules of the form $A \to \varepsilon$
   - A must not be the start variable
   - Then for every rule with A on RHS, add new rule with A deleted
     - E.g., If $R \to uAv$ is a rule, add $R \to uv$
   - Must cover all combinations if A appears more than once in a RHS
     - E.g., if $R \to uAvAw$ is a rule, add 3 rules: $R \to uvAw$, $R \to uAvw$, R -> uvw
3. Remove all "unit" rules of the form $A \to B$
   - Then, for every rule $B \to u$, add rule $A \to u$
4. Split up rules with RHS longer than length 2
   - E.g., $A \to wxyz$ becomes $A \to wB$, $B \to xC$, $C \to yz$
5. Replace all terminals on RHS with new rule
   - E.g., for above, add $W \to w$, $X \to x$, $Y \to y$, $Z \to z$

$$
\begin{aligned}
S_0 &\to \boxed{ASA} \mid \boxed{\mathtt{a}B} \mid \mathtt{a} \mid SA \mid AS \\
S &\to ASA \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \\
A &\to \mathtt{b} \mid ASA \mid \mathtt{a}B \mid \mathtt{a} \mid SA \mid AS \\
B &\to \mathtt{b}
\end{aligned}
$$

$$
\begin{aligned}
S_0 &\to \boxed{AA_1} \mid \boxed{UB} \mid \mathtt{a} \mid SA \mid AS \\
S &\to AA_1 \mid UB \mid \mathtt{a} \mid SA \mid AS \\
A &\to \mathtt{b} \mid AA_1 \mid UB \mid \mathtt{a} \mid SA \mid AS \\
A_1 &\to \boxed{SA} \\
\boxed{U} &\boxed{\to \mathtt{a}} \\
B &\to \mathtt{b}
\end{aligned}
$$

Thm: A_{CFG} is a decidable language

$$A_{\mathsf{CFG}} = \{\langle G, w\rangle |\ G \text{ is a CFG that generates string } w\}$$

Proof: create the decider:

$S =$ "On input $\langle G, w\rangle$, where G is a CFG and w is a string:

1. Convert G to an equivalent grammar in Chomsky normal form.
2. List all derivations with $2n - 1$ steps, where n is the length of w; except if $n = 0$, then instead list all derivations with one step.
3. If any of these derivations generate w, *accept*; if not, *reject*."

We first need to prove this is true for all CFGs!

Termination argument:
Step 1: any CFG has only a finite # rules
Step 2: $2n$-1 = finite # of derivations to check
Step 3: only 1 step

Thm: E_{CFG} is a decidable language

$$E_{\mathsf{CFG}} = \{\langle G\rangle \mid G \text{ is a } \boxed{\mathsf{CFG}} \text{ and } L(G) = \emptyset\}$$

Recall:

$$E_{\mathsf{DFA}} = \{\langle A\rangle \mid A \text{ is a } \boxed{\mathsf{DFA}} \text{ and } L(A) = \emptyset\}$$

$T =$ "On input $\langle A\rangle$, where A is a DFA:

1. Mark the start state of A.
2. Repeat until no new states get marked:
3. Mark any state that has a transition coming into it from any state that is already marked.
4. If no accept state is marked, *accept*; otherwise, *reject*."

"Reachability" (of accept state from start state) algorithm

Can we compute "reachability" for a CFG?

Thm: E_{CFG} is a decidable language

$E_{\mathsf{CFG}} = \{\langle G \rangle | \ G \text{ is a CFG and } L(G) = \emptyset\}$

Proof: create **decider** that calculates reachability for grammar G

- Go backwards, start from **terminals**, to avoid getting stuck in looping rules

$R =$ "On input $\langle G \rangle$, where G is a CFG:

1. Mark all terminal symbols in G.
2. Repeat until no new variables get marked:
3. Mark any variable A where G has a rule $A \rightarrow U_1 U_2 \cdots U_k$ and each symbol $U_1, \ldots, U_k$ has already been marked.
4. If the start variable is not marked, *accept*; otherwise, *reject*."

Loop marks 1 new variable on each iteration or stops: it eventually terminates because there are a finite # of variables

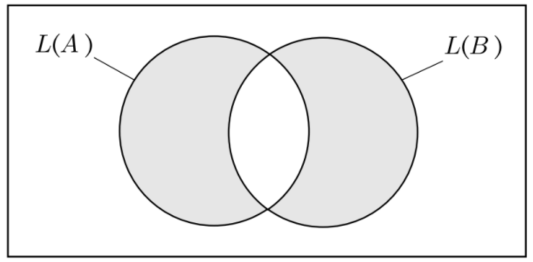
Termination argument?

Thm: EQ_{CFG} is a decidable language?

$EQ_{\text{CFG}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

Recall: $EQ_{\text{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$

- Used Symmetric Difference



$L(C) = \emptyset \text{ iff } L(A) = L(B)$

  - where C = complement, union, intersection of machines A and B
- Can't do this for CFLs!
  - Intersection and complement are not closed for CFLs!!!

Intersection of CFLs is Not Closed!

Proof (by contradiction), Assume **intersection is closed for CFLs**

- Then **intersection of these CFLs should be a CFL:**

$$A = \{\mathtt{a}^m\mathtt{b}^n\mathtt{c}^n \mid m, n \geq 0\}$$

$$B = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^m \mid m, n \geq 0\}$$

- But $A \cap B = \{\mathtt{a}^n\mathtt{b}^n\mathtt{c}^n \mid n \geq 0\}$

- … **which is not a CFL!** (So we have a contradiction)

Complement of a CFL is not Closed!

- Assume CFLs closed under complement, then:

$$\text{if } G_1 \text{ and } G_2 \text{ context-free}$$

$$\overline{L(G_1)} \text{ and } \overline{L(G_2)} \text{ context-free}$$ From the assumption

$$\overline{L(G_1)} \cup \overline{L(G_1)} \text{ context-free}$$ Union of CFLs is closed

$$\overline{\overline{L(G_1)} \cup \overline{L(G_1)}} \text{ context-free}$$ From the assumption

$$L(G_1) \cap L(G_2) \text{ context-free}$$ DeMorgan's Law!

But intersection is not closed for CFLS (prev slide)

Thm: EQ_{CFG} is a decidable language?

$EQ_{\mathsf{CFG}} = \{\langle G, H\rangle |\ G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

- No! 
    - There's no algorithm to decide whether two grammars are equivalent!

- **It's not recognizable either!** (Can't create <u>any</u> TM to do this!!!)
    - (details later)

- I.e., **this is an <u>impossible computation</u>!**

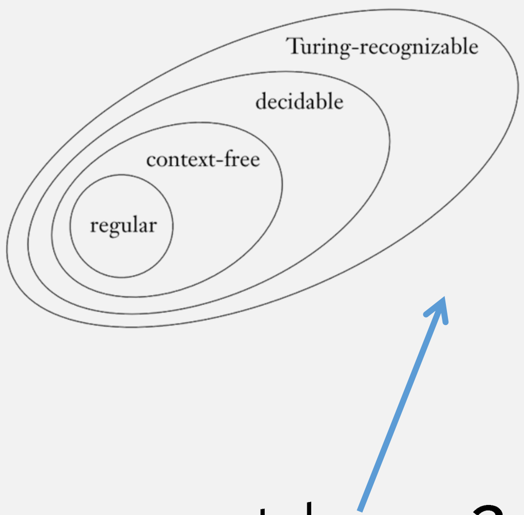
Summary Algorithms About CFLs

- $A_{\text{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}$
  - **Decider:** Convert grammar to Chomsky Normal Form
  - Then **check all possible derivations up to length 2|w| - 1 steps**

- $E_{\text{CFG}} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$
  - **Decider:** Compute "reachability" of start variable from terminals

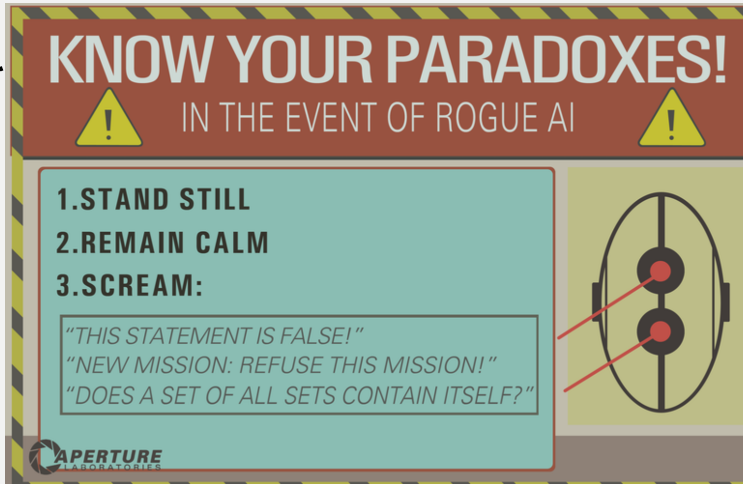
- $EQ_{\text{CFG}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$
  - **We couldn't prove that this is decidable!**
  - (So you cant use this theorem when creating another decider)

The Limits of Turing Machines?



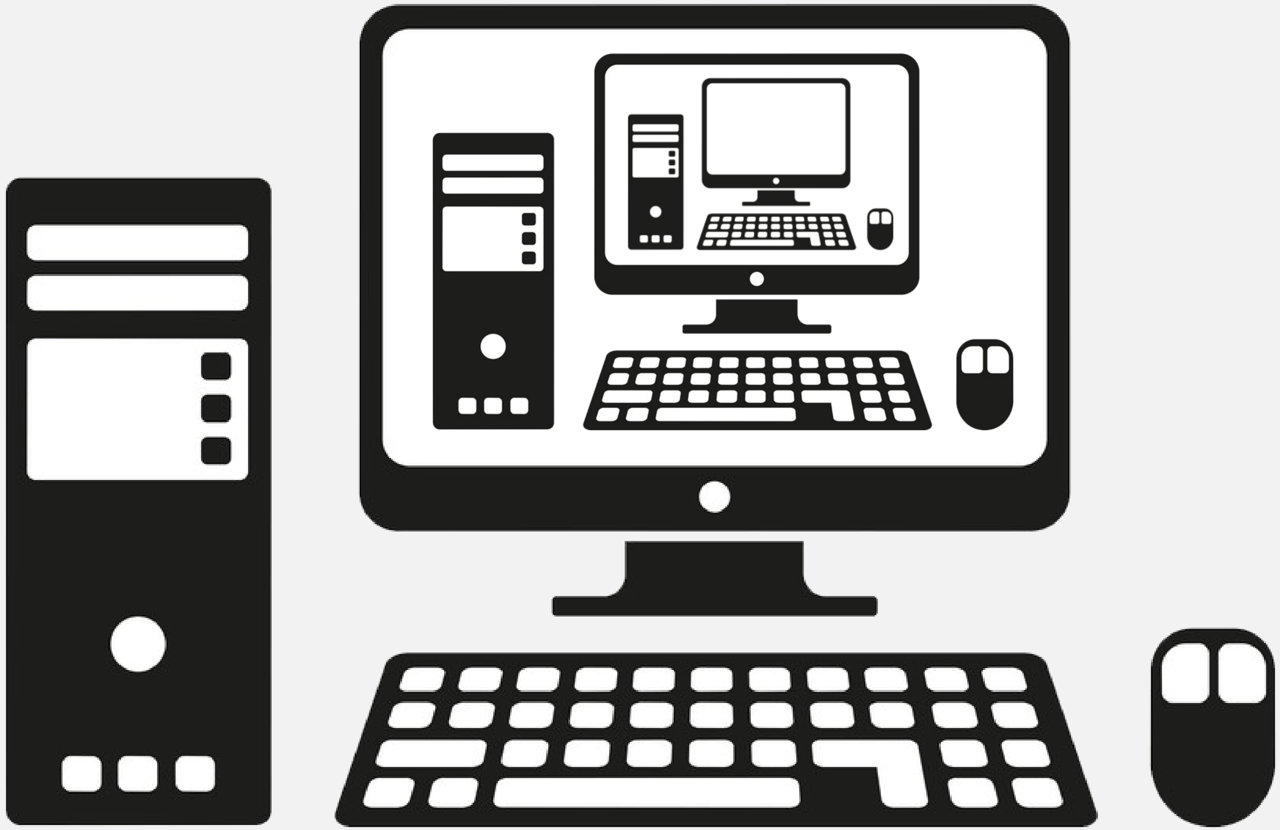
- **TMs represent all possible "computations"**
  - I.e., **any (`Python`, `Java`, …) program you write is a TM**
- But **some things are not computable?** I.e., some langs are out here **?**
- **To test the limit of a computational model, we can see …**
  … what it can compute about other computational model
  - Thought: **Is there a decider (algorithm) to determine whether a TM is an decider?**

Hmmm, this doesn't feel right …



Next time: Is A_{TM} decidable?

$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$



Check-in Quiz 11/8

On gradescope