CS420

Reducibility

Tuesday November 15, 2022

Announcements

- HW 8 in
  - ~~Due Mon 11/14 11:59pm~~

- HW 9 out
  - Due Mon 11/21 11:59pm

Last Time: Undecidability Proofs

- We proved $A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$ **undecidable …**
- **… by** contradiction:
  - **Use hypothetical** A_{TM} **decider to create an <u>impossible</u> decider "D"!**

- Step # 1: **coming up with "D" --- <u>hard!</u>**
  - Need to invent **diagonalization**

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\cdots$	$\langle D\rangle$
M_1	*accept*	*reject*	*accept*	*reject*		*accept*
M_2	*accept*	*accept*	*accept*	*accept*	$\cdots$	*accept*
M_3	*reject*	*reject*	*reject*	*reject*		*reject*
M_4	*accept*	*accept*	*reject*	*reject*		*accept*
$\vdots$			$\vdots$		$\ddots$	
D	*reject*	*reject*	*accept*	*accept*		?

- Step # 2: **"reduce"** A_{TM} **to the "D" problem --- <u>easier</u>!**

- From now on: **undecidability proofs <u>only need</u> <u>step # 2</u>!**
  - And we now **have two "impossible" problems to choose from**

Last Time: The Halting Problem

$$\boxed{HALT_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}}$$

Thm: $HALT_{\mathsf{TM}}$ is undecidable

Proof, by contradiction:

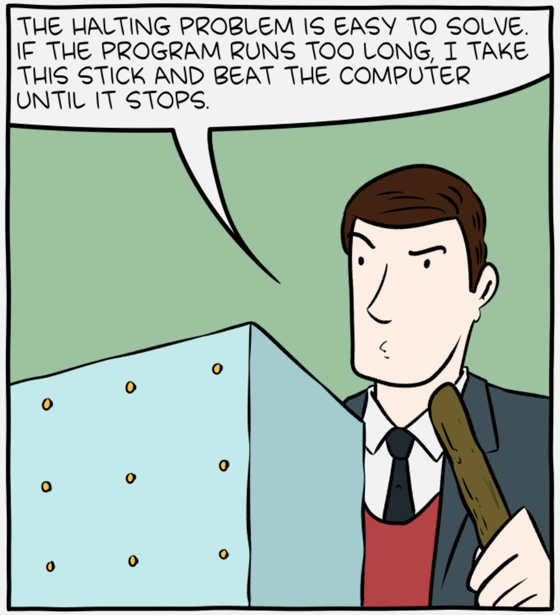
- Assume: $HALT_{\mathsf{TM}}$ has *decider* R; use it to create decider for A_{TM}:

$$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$$

- …

contradiction

- But A_{TM} is undecidable and has no decider!



What if Alan Turing had been an engineer?

Last Time: The Halting Problem

$HALT_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$

Thm: $HALT_{\mathsf{TM}}$ is undecidable

Proof, by contradiction:

Using our hypothetical $HALT_{\mathsf{TM}}$ decider R

- Assume: $HALT_{\mathsf{TM}}$ has *decider* R; use it to create decider for A_{TM}:

$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$

S = "On input $\langle M, w\rangle$, an encoding of a TM M and a string w:

1. Run TM R on input $\langle M, w\rangle$.
2. If R rejects, *reject*. ← This means M loops on input w
3. If R accepts, simulate M on w until it halts. ← This step always halts
4. If M has accepted, *accept*; if M has rejected, *reject*."

Termination argument:
Step 1: R is a decider so always halts
Step 3: M always halts because R said so

Undecidability Proof Technique #1:
Reduce (directly) **from** A_{TM}

Last Time: The Halting Problem

$HALT_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$

Thm: $HALT_{\mathsf{TM}}$ **is undecidable**

Proof, by contradiction:

- Assume: $HALT_{\mathsf{TM}}$ **has *decider* R; use it to create decider for A_{TM}:**

  $A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$

  S = "On input $\langle M, w\rangle$, an encoding of a TM M and a string w:
  1. Run TM R on input $\langle M, w\rangle$.
  2. If R rejects, *reject*.
  3. If R accepts, simulate M on w until it halts.
  4. If M has accepted, *accept*; if M has rejected, *reject*."

  Now we have three "impossible" deciders to choose from

- But A_{TM} **is undecidable!** I.e., **this decider does not exist!**
  - **So $HALT_{\mathsf{TM}}$ is also undecidable!**

Interlude: Reducing from $HALT_{\mathrm{TM}}$

A practical thought experiment …
… about compiler optimizations

Your compiler changes your program!

```
If TRUE then A else B   ⇨   A

           1 + 2 + 3    ⇨   6
```

Compiler Optimizations

Optmization - docs

- -O0
  - No optmization, faster compilation time, better for debugging builds.
- -O2
- -O3
  - Higher level of optmization. Slower compile-time, better for production builds.
- -OFast
  - Enables higher level of optmization than (-O3). It enables lots of flags as can be seen src (-ffloat-store, -ffsast-math, -ffinite-math-only, -O3 …)
- -finline-functions
- -m64
- -funroll-loops
- -fvectorize
- -fprofile-generate

Types of optimization [edit]

Techniques used in optimization can be broken up among various *scopes* which can affect anything from a single statement to the entire program. Generally speaking, locally scoped techniques are easier to implement than global ones but result in smaller gains. Some examples of scopes include:

Peephole optimizations

These are usually performed late in the compilation process after machine code has been generated. This form of optimization examines a few adjacent instructions (like "looking through a peephole" at the code) to see whether they can be replaced by a single instruction or a shorter sequence of instructions.[2] For instance, a multiplication of a value by 2 might be more efficiently executed by left-shifting the value or by adding the value to itself (this example is also an instance of strength reduction).

Local optimizations

These only consider information local to a basic block.[3] Since basic blocks have no control flow, these optimizations need very little analysis, saving time and reducing storage requirements, but this also means that no information is preserved across jumps.

Global optimizations

These are also called "intraprocedural methods" and act on whole functions.[3] This gives them more information to work with, but often makes expensive computations necessary. Worst case assumptions have to be made when function calls occur or global variables are accessed because little information about them is available.

Loop optimizations

These act on the statements which make up a loop, such as a *for* loop, for example loop-invariant code motion. Loop optimizations can have a significant impact because many programs spend a large percentage of their time inside loops.[4]

Prescient store optimizations

These allow store operations to occur earlier than would otherwise be permitted in the context of threads and locks. The process needs some way of knowing ahead of time what value will be stored by the assignment that it should have followed. The purpose of this relaxation is to allow compiler optimization to perform certain kinds of code rearrangement that preserve the semantics of properly synchronized programs.[5]

Interprocedural, whole-program or link-time optimization

These analyze all of a program's source code. The greater quantity of information extracted means that optimizations can be more effective compared to when they only have access to local information, i.e. within a single function. This kind of optimization can also allow new techniques to be performed. For instance, function inlining, where a call to a function is replaced by a copy of the function body.

Machine code optimization and object code optimizer

These analyze the executable task image of the program after all of an executable machine code has been linked. Some of the techniques that can be applied in a more limited scope, such as macro compression which saves space by collapsing common sequences of instructions, are more effective when the entire executable task image is available for analysis.[6]

The Optimal Optimizing Compiler

"Full Employment" Theorem

Thm: **The Optimal (C++) Optimizing Compiler does not exist**

Proof, by contradiction:

Assume: ***OPT* is the Perfect Optimizing Compiler**

Use it **to create $HALT_{TM}$ decider** (**accept**s <*M*,*w*> if *M* halts with *w*, else **reject**s):

S = **On input** <*M*, *w*>, where *M* is **C++** program and *w* is string:

- If *OPT*(*M*) == `for(;;)`
    a) Then **Reject**
    b) Else **Accept**

In computer science and mathematics, a **full employment theorem** is a term used, often humorously, to refer to a theorem which states that no algorithm can optimally perform a particular task done by some class of professionals. The name arises because such a theorem ensures that there is endless scope to keep discovering new techniques to improve the way at least some specific task is done.

For example, the *full employment theorem for compiler writers* states that there is no such thing as a provably perfect size-optimizing compiler, as such a proof for the compiler would have to detect non-terminating computations and reduce them to a one-instruction infinite loop. Thus, the existence of a provably perfect size-optimizing compiler would imply a solution to the halting problem, which cannot exist. This also implies that there may always be a better compiler since the proof that one has the best compiler cannot exist. Therefore, compiler writers will always be able to speculate that they have something to improve.

Summary: The Limits of Algorithms

- $A_{\mathsf{DFA}} = \{\langle B, w\rangle |\ B \text{ is a DFA that accepts input string } w\}$ Decidable
- $A_{\mathsf{CFG}} = \{\langle G, w\rangle |\ G \text{ is a CFG that generates string } w\}$ Decidable
- $A_{\mathsf{TM}} = \{\langle M, w\rangle |\ M \text{ is a TM and } M \text{ accepts } w\}$ **Undecidable**
- $HALT_{\mathsf{TM}} = \{\langle M, w\rangle |\ M \text{ is a TM and } M \text{ halts on input } w\}$ **Undecidable**
- $E_{\mathsf{DFA}} = \{\langle A\rangle |\ A \text{ is a DFA and } L(A) = \emptyset\}$ Decidable
- $E_{\mathsf{CFG}} = \{\langle G\rangle |\ G \text{ is a CFG and } L(G) = \emptyset\}$ Decidable
- next $E_{\mathsf{TM}} = \{\langle M\rangle |\ M \text{ is a TM and } L(M) = \emptyset\}$ **Undecidable**

Reducibility: Modifying the TM

$E_{\mathsf{TM}} = \{\langle M \rangle | \ M \text{ is a TM and } L(M) = \emptyset\}$

<u>Thm</u>: E_{TM} is undecidable

<u>Proof</u>, by contradiction:

- Assume E_{TM} has *decider* R; use it to create *decider* for A_{TM}:

$S =$ "On input $\langle M, w \rangle$, an encoding of a TM M and a string w:

First, construct M_1

- Run R on input $\langle M_1 \rangle$

Note: M_1 is <u>only</u> used as arg to R; <u>we never run it!</u>

- If R accepts, *reject* (because it means $\langle M \rangle$ doesn't accept w)
- if R rejects, then *accept* ($\langle M \rangle$ accepts w)

- <u>Idea</u>: Wrap $\langle M \rangle$ in a new TM that <u>can *only* accept w</u>:

$M_1 =$ "On input x:
1. If $x \neq w$, *reject*.
2. If $x = w$, run M on input w and *accept* if M does."

Input not w, always reject

Input is w, maybe accept

M_1 accepts w if M does

Reducibility: Modifying the TM

$E_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

Thm: E_{TM} is undecidable

Proof, by contradiction:

This decider for A_{TM} cannot exist!

- Assume E_{TM} has *decider* R; use it to create *decider* for A_{TM}:

$S =$ "On input $\langle M, w \rangle$, an encoding of a TM M and a string w:

First, construct M_1

- Run R on input $\langle M_{\boxed{1}} \rangle$
- If R accepts, *reject* (because it means $\langle M \rangle$ doesn't accept $\boxed{w}$)
- if R rejects, then $\boxed{\textit{accept}}$ ($\langle M \rangle$ accepts $\boxed{w}$)

- Idea: Wrap $\langle M \rangle$ in a new TM that can *only* accept *w*:

$M_1 =$ "On input x:
1. If $x \neq w$, *reject*.
2. If $x = w$, run M on input w and *accept* if M does."

Summary: The Limits of Algorithms

- $A_{\mathsf{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$ Decidable
- $A_{\mathsf{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}$ Decidable
- $A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$ **Undecidable**
- $E_{\mathsf{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$ Decidable
- $E_{\mathsf{CFG}} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$ Decidable
- $E_{\mathsf{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$ **Undecidable**
- $EQ_{\mathsf{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$ Decidable
- $EQ_{\mathsf{CFG}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$ **Undecidable**
- $EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$ **Undecidable**

needs

next

Reduce to something else: EQ_{TM} is undecidable

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Proof, by contradiction:

- Assume: EQ_{TM} has *decider* R; use it to create *decider* for $\cancel{A_{\mathsf{TM}}}$:

$E_{\mathsf{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

$S =$ "On input $\langle M\rangle$, where M is a TM:

1. Run R on input $\langle M, M_1\rangle$, where M_1 is a TM that rejects all inputs.
2. If R accepts, *accept*; if R rejects, *reject*."

Reduce to something else: EQ_{TM} is undecidable

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Proof, by contradiction:

- Assume: EQ_{TM} has *decider* R; use it to create *decider* for E_{TM}:

$= \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

$S =$ "On input $\langle M\rangle$, where M is a TM:

1. Run R on input $\langle M, M_1\rangle$, where M_1 is a TM that rejects all inputs.
2. If R accepts, *accept*; if R rejects, *reject*."

- But E_{TM} is undecidable!

Summary: Undecidability Proof Techniques

$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$

Reduce

- Proof Technique #1:
  - **Use hypothetical decider to implement impossible A_{TM} decider**
  - Example **Proof:** $HALT_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$
- Proof Technique #2:
  - **Use hypothetical decider to implement impossible A_{TM} decider**
  - But **first modify the input M**
  - Example **Proof:** $E_{\mathsf{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

Reduce

- Proof Technique #3:
  - **Use hypothetical decider to implement non-A_{TM} impossible decider**
  - Example **Proof:** $EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Summary: Decidability and Undecidability

- $A_{\mathsf{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$ Decidable
- $A_{\mathsf{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}$ Decidable
- $A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$ **Undecidable**
- $E_{\mathsf{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$ Decidable
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- $E_{\mathsf{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$ **Undecidable**
- $EQ_{\mathsf{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$ Decidable
- $EQ_{\mathsf{CFG}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$ **Undecidable**
- $EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$ **Undecidable**

Also Undecidable …

next

- $REGULAR_{\mathrm{TM}} = \{<M> \mid M$ is a TM and $L(M)$ is a regular language$\}$

Undecidability Proof Technique #2:
Modify input TM M

Thm: $REGULAR_{\mathsf{TM}}$ is undecidable

$REGULAR_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is a regular language}\}$

Proof, by contradiction:

- Assume: $REGULAR_{\mathsf{TM}}$ has *decider* R; use it to create *decider* for A_{TM}:

$S =$ "On input $\langle M, w \rangle$, an encoding of a TM M and a string w:

- First, construct M_2 (??)
- Run R on input $\langle M_2 \rangle$
- If R accepts, *accept*; if R rejects, *reject*

Want: $L(M_2)$ =
- **regular**, if M accepts w
- **nonregular**, if M does not accept w

Thm: $REGULAR_{\mathsf{TM}}$ is undecidable (continued)

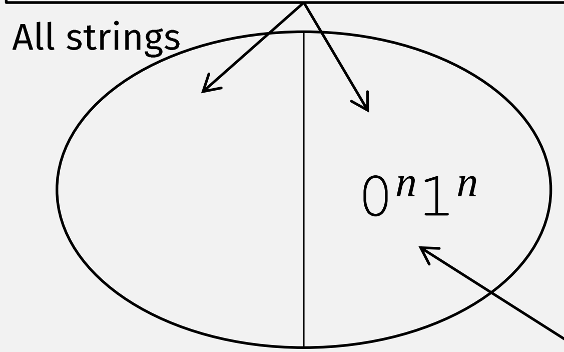
$REGULAR_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is a regular language}\}$

M_2 = "On input x:
1. If x has the form 0^n1^n, *accept*.
2. If x does not have this form, run M on input w and *accept* if M accepts w."

Always accept strings 0^n1^n
$L(M_2)$ = **nonregular**, so far

If M accepts w,
accept everything else,
so $L(M_2) = \Sigma^* =$ **regular**

if M does not accept w, M_2 accepts all strings (**regular** lang)



if M accepts w, M_2 accepts this **nonregular** lang

Want: $L(M_2)$ =
- **regular**, if M accepts w
- **nonregular**, if M does not accept w

Also Undecidable …

Seems like no algorithm can compute **anything** about language of TMs, i.e., about programs …

- $REGULAR_{\mathrm{TM}} = \{<M> \mid M \text{ is a TM and } L(M) \text{ is a regular language}\}$
- $CONTEXTFREE_{\mathrm{TM}} = \{<M> \mid M \text{ is a TM and } L(M) \text{ is a CFL}\}$
- $DECIDABLE_{\mathrm{TM}} = \{<M> \mid M \text{ is a TM and } L(M) \text{ is a decidable language}\}$
- $FINITE_{\mathrm{TM}} = \{<M> \mid M \text{ is a TM and } L(M) \text{ is a finite language}\}$

An Algorithm About Program Behavior?

```
main()
{
    printf("hello, world\n");
}
```

Write a program that,
given another program as its argument,
returns TRUE if that argument prints
"**`Hello, World!`**"

TRUE

Fermat's Last Theorem
(unknown for ~350 years,
solved in 1990s)

```
main()
{
   If x^n + y^n = z^n, for any integer n > 2
      printf("hello, world\n");
}
```

$$\boxed{x^n + y^n = z^n, \text{ for any integer } n > 2}$$

Write a program that,
given another program as its argument,
returns TRUE if that argument prints
"`Hello, World!`"

?????

Also Undecidable …

Seems like no algorithm can compute **anything** about Turing Machines, i.e., about programs …

- $REGULAR_{\mathrm{TM}}$ = {<M> | M is a TM and $L(M)$ is a regular language}
- $CONTEXTFREE_{\mathrm{TM}}$ = {<M> | M is a TM and $L(M)$ is a CFL}
- $DECIDABLE_{\mathrm{TM}}$ = {<M> | M is a TM and $L(M)$ is a decidable language}
- $FINITE_{\mathrm{TM}}$ = {<M> | M is a TM and $L(M)$ is a finite language}
- …

Rice's Theorem

- $ANYTHING_{\mathrm{TM}}$ = {<M> | M is a TM and "… **anything** …" about $L(M)$}

Rice's Theorem: $ANYTHING_{\text{TM}}$ is Undecidable

$ANYTHING_{\text{TM}}$ = {<M> | M is a TM and … **anything** … about $L(M)$}

- "… **Anything** …", more precisely:
  - For any M_1, M_2, if $L(M_1) = L(M_2)$ …
  - … then $M_1 \in ANYTHING_{\text{TM}} \Leftrightarrow M_2 \in ANYTHING_{\text{TM}}$

- Also, "… **Anything** …"must be "non-trivial":
  - $ANYTHING_{\text{TM}}$!= {}
  - $ANYTHING_{\text{TM}}$!= set of all TMs

Rice's Theorem: $ANYTHING_{\mathsf{TM}}$ is Undecidable

$ANYTHING_{\mathsf{TM}}$ = {<M> | M is a TM and … **anything** … about $L(M)$}

Proof by contradiction

- Assume some language satisfying $ANYTHING_{\mathsf{TM}}$ has a decider R.
  - Since $ANYTHING_{\mathsf{TM}}$ is non-trivial, then there exists $M_{ANY} \in ANYTHING_{\mathsf{TM}}$
  - Where R accepts M_{ANY}
- Use R to create decider for A_{TM}:

On input <M, w>:

- Create M_w:

  M_w = on input x:
  - Run M on w
    - If M rejects w: reject x
    - If M accepts w:
      Run M_{ANY} on x and accept if it accepts, else reject

  If M accepts w: M_w = M_{ANY}
  If M doesn't accept w: M_w accepts nothing

- Run R on M_w
  - If it accepts, then M_w = M_{ANY}, so M accepts w, so accept
  - Else reject

These two cases must be <u>different</u>, (so R can distinguish when M accepts w)

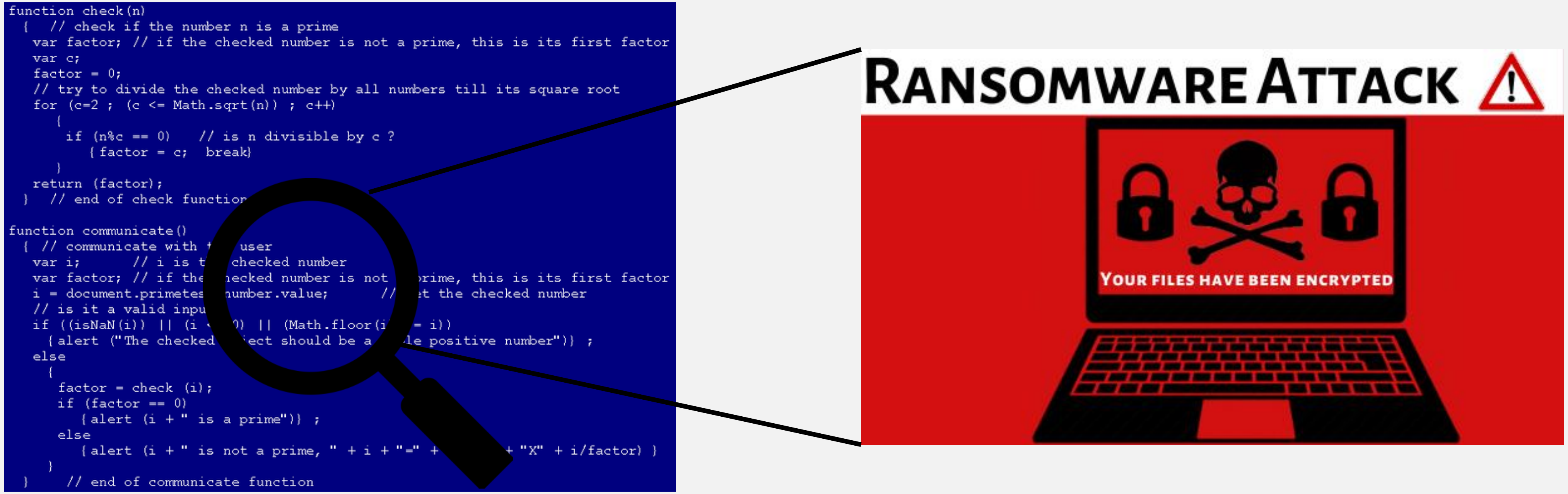
Wait! What if the TM that accepts nothing is in $ANYTHING_{\mathsf{TM}}$!

Proof still works! Just use the complement of $ANYTHING_{\mathsf{TM}}$ instead!

Rice's Theorem Implication

{<M> | M is a TM that installs malware}

Undecidable!
(by Rice's Theorem)



$A_{\mathsf{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$ Decidable

$A_{\mathsf{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}$ Decidable

$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$ **Undecidable**

- In hindsight, of course a restricted TM (a **decider**) shouldn't be able to simulate unrestricted TM (a **recognizer**)

- But could a restricted TM simulate an even more restricted TM?
  - Next time

Check-in Quiz 11/15

On gradescope