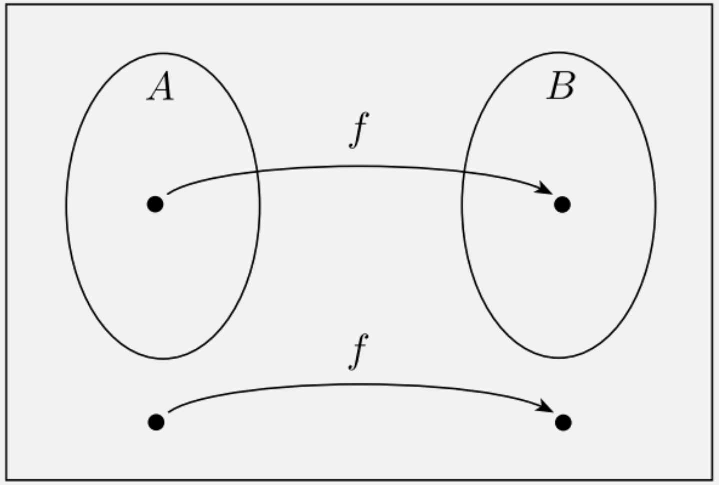
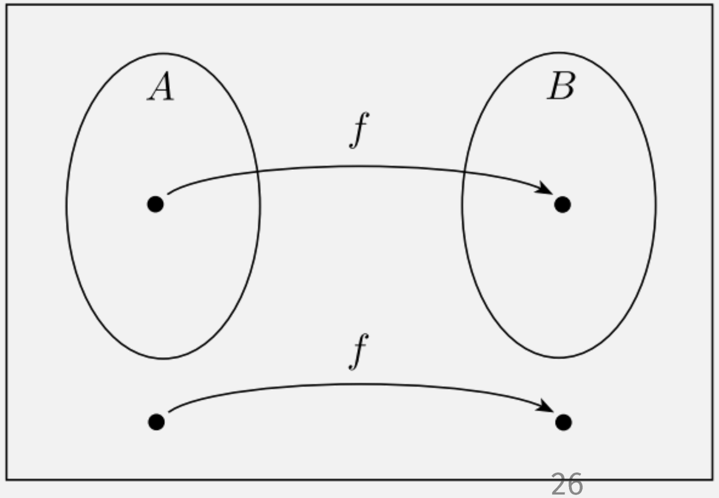
Mapping Reducibility

Wednesday, April 7, 2021



Announcements

- HW 8 due Sun 4/11 11:59pm EST
- HW 9 out
  - Due Sun 4/18 11:59pm EST
  - Ch5 material (starting today)



Last time: “Reduced” A_{TM} to $HALT_{\mathsf{TM}}$

$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$

↓

$HALT_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$

Thm: $HALT_{\mathsf{TM}}$ is undecidable

Proof, by contradiction:

- Assume $HALT_{\mathsf{TM}}$ has *decider* R; use to create A_{TM} *decider*:

S = “On input $\langle M, w\rangle$, an encoding of a TM M and a string w:

1. Run TM R on input $\langle M, w\rangle$. ← Use R to first check if M will loop on w
2. If R rejects, *reject*.
3. If R accepts, simulate M on w until it halts. ← Then run M on w knowing it won't loop
4. If M has accepted, *accept*; if M has rejected, *reject*.”

- Contradiction: A_{TM} is undecidable and has no decider!

Today: Formalize “reduction” and “reducibilty”

Last time: $REGULAR_{\mathsf{TM}}$ is undecidable

$REGULAR_{\mathsf{TM}} = \{\langle M \rangle | \ M \text{ is a TM and } L(M) \text{ is a regular language}\}$

Proof, by contradiction:

- Assume $REGULAR_{\mathsf{TM}}$ has *decider* R; use to create A_{TM} *decider*:

$S =$ "On input $\langle M, w\rangle$, an encoding of a TM M and a string w:

- First, construct M_2 (see below, and next slide)
- Run R on input $\langle M_2 \rangle$
- If R accepts, *accept*; if R rejects, *reject*

Important: M_2 is never run; only used as an arg

Want: $L(M_2)$ =
- regular, if M accepts w
- nonregular, if M does not accept w

Thm: $REGULAR_{\text{TM}}$ is undecidable (continued)

$REGULAR_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) \text{ is a regular language}\}$

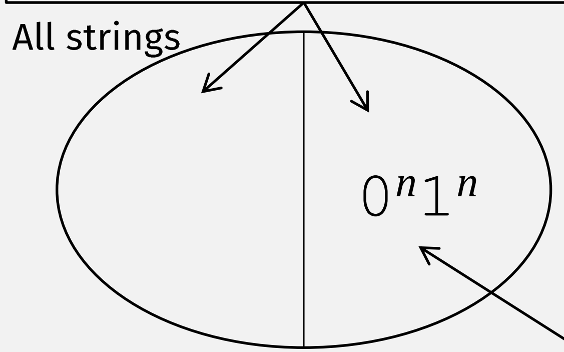
M_2 = "On input x:

1. If x has the form 0^n1^n, *accept*.
2. If x does not have this form, run M on input w and *accept* if M accepts w."

Always accept strings 0^n1^n
$L(M_2)$ = nonregular, so far

If M accepts w,
accept everything else,
so $L(M_2) = \Sigma^*$ = regular

if M does not accept w, M_2 accepts all strings (regular lang)



if M accepts w, M_2 accepts this non-regular lang

Want: $L(M_2)$ =
- regular, if M accepts w
- nonregular, if M does not accept w

Reducing to non-A_{TM} language

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

<u>Thm</u>: EQ_{TM} is undecidable

<u>Proof</u>, by contradiction:

$E_{\mathsf{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

- Assume EQ_{TM} has *decider* R; use to create ~~A_{TM}~~ E_{TM} *decider*:

S = "On input $\langle M\rangle$, where M is a TM:

1. Run R on input $\langle M, M_1\rangle$, where M_1 is a TM that rejects all inputs.
2. If R accepts, *accept*; if R rejects, *reject*."

Reducing to non-A_{TM} language

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle |\ M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

<u>Thm</u>: EQ_{TM} is undecidable

<u>Proof</u>, by contradiction:

$E_{\mathsf{TM}} = \{\langle M\rangle |\ M \text{ is a TM and } L(M) = \emptyset\}$

- Assume EQ_{TM} has *decider* R; use to create ~~A_{TM}~~ E_{TM} *decider*:

~~S = "On input $\langle M\rangle$, where M is a TM:~~

~~**1.** Run R on input $\langle M, M_1\rangle$, where M_1 is a TM that rejects all inputs.~~

~~**2.** If R accepts, *accept*; if R rejects, *reject*."~~

- <u>Contradiction</u>: E_{TM} is undecidable!

Summary

- $A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$ — Decidable
- $A_{\text{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}$ — Decidable
- $A_{\text{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$ — **Undecidable**
- $E_{\text{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$ — Decidable
- $E_{\text{CFG}} = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$ — Decidable
- $E_{\text{TM}} = \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$ — **Undecidable**
- $EQ_{\text{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$ — Decidable
- $EQ_{\text{CFG}} = \{\langle G, H\rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$ — **Undecidable**
- $EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$ — **Undecidable**

Observation:
Can we decide anything about Turing Machines, i.e., about programs?

Can't decide anything about TMs?

- $REGULAR_{TM}$ = {<M> | M is a TM and $L(M)$ is a regular language} **Undecidable**

HW9
- $CONTEXTFREE_{TM}$ = {<M> | M is a TM and $L(M)$ is a CFL} **Undecidable**

- $DECIDABLE_{TM}$ = {<M> | M is a TM and $L(M)$ is a decidable language} **Undecidable**

- $FINITE_{TM}$ = {<M> | M is a TM and $L(M)$ is a finite language} **Undecidable**

- …

Undecidable:

Rice's Theorem

HW9
- $ANYTHING_{TM}$ = {<M> | M is a TM and "something something" about $L(M)$}

Today: Computable Functions

- Needed to formalize the notion of “reducibility”

Flashback: A_{NFA} is a decidable language

$A_{\text{NFA}} = \{\langle B, w\rangle \mid B \text{ is an NFA that accepts input string } w\}$

Decider (i.e., "run" function) for A_{NFA} :

N = "On input $\langle B, w\rangle$, where B is an NFA and w is a string:

1. Convert NFA B to an equivalent DFA C, using the procedure for this conversion given in Theorem 1.39.

2. Run TM M on input $\langle C, w\rangle$.

3. If M accepts, *accept*; otherwise, *reject*."

We said this NFA -> DFA algorithm is a TM, but it doesn't accept/reject?

More generally, we've been saying "**programs = TMs**", but programs do more than accept/reject?

Computable Functions

- A TM that, instead of accept/reject, "outputs" final tape contents

DEFINITION 5.17

A function $f : \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

- Example 1: All arithmetic operations

- Example 2: Converting between machines, like DFA -> NFA
  - E.g., adding states, changing transitions, wrapping TM in TM, etc.

Mapping Reducibility

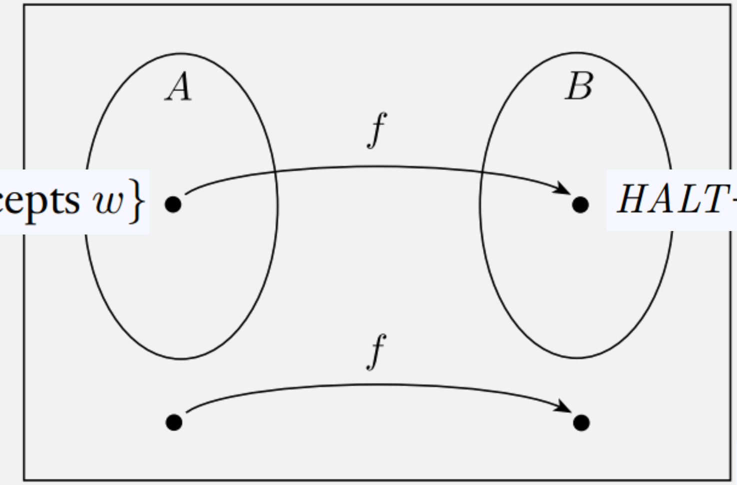
DEFINITION 5.20

Language A is ***mapping reducible*** to language B, written $A \leq_m B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \iff f(w) \in B.$$

The function f is called the ***reduction*** from A to B.

$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$



$HALT_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$

DEFINITION 5.17

A function $f\colon \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

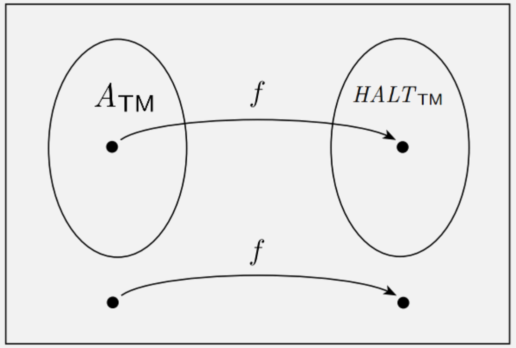
Thm: A_{TM} is mapping reducible to $HALT_{\mathsf{TM}}$

$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$

$HALT_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$

- To show: $A_{\mathsf{TM}} \leq_{\mathrm{m}} HALT_{\mathsf{TM}}$
- Want: computable fn f : $\langle M, w\rangle$ → $\langle M', w'\rangle$ where:

$\langle M, w\rangle \in A_{\mathsf{TM}}$ if and only if $\langle M', w'\rangle \in HALT_{\mathsf{TM}}$



The following machine F computes a reduction f.

$F =$ "On input $\langle M, w\rangle$:

1. Construct the following machine M'.
   $M' =$ "On input x:
   1. Run M on x.
   2. If M accepts, *accept*.
   3. If M rejects, enter a loop."
2. Output $\langle M', w\rangle$."

Converts M to M'

M accepts w if and only if M' halts on w

Output new M'

M' is like M, except it always loops when it doesn't accept

DEFINITION 5.20

Language A is ***mapping reducible*** to language B, written $A \leq_{\mathrm{m}} B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \iff f(w) \in B.$$

The function f is called the ***reduction*** from A to B.

DEFINITION 5.17

A function $f\colon \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

How is mapping reducibility useful?

Thm: If $A \leq_m B$ and B is decidable, then A is decidable.

Has a decider

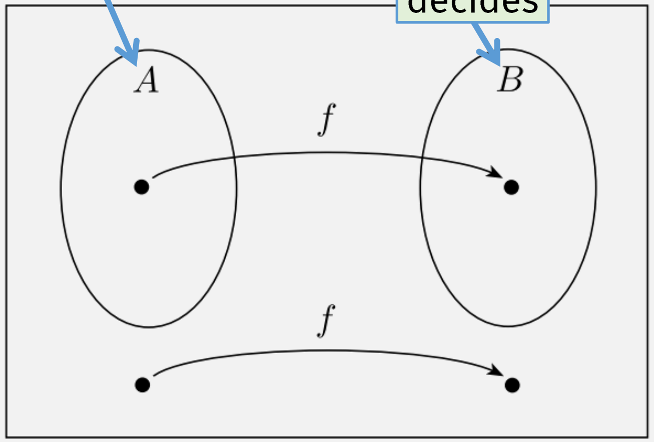
PROOF We let M be the decider for B and f be the reduction from A to B. We describe a decider N for A as follows.

$N =$ "On input w:

1. Compute $f(w)$.
2. Run M on input $f(w)$ and output whatever M outputs."

decides

decides



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Language A is ***mapping reducible*** to language B, written $A \leq_m B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \iff f(w) \in B.$$

The function f is called the ***reduction*** from A to B.

Coro: If $A \leq_m B$ and A is undecidable, then B is undecidable.

- Proof by contradiction.
- Assume B is decidable.
- Then A is decidable (by the previous thm).
- Contradiction: we already said A is undecidable

If $A \leq_m B$ and B is decidable, then A is decidable.

Summary: Mapping Reducibility Theorems

- If $A \leq_m B$ and B is decidable, then A is decidable.

Known

Unknown

- If $A \leq_m B$ and A is undecidable, then B is undecidable.

Alternate Proof: The Halting Problem

$HALT_{\mathsf{TM}}$ is undecidable

- If $A \leq_m B$ and A is undecidable, then B is undecidable.
- $A_{\mathsf{TM}} \leq_m HALT_{\mathsf{TM}}$
- Since A_{TM} is undecidable, then $HALT_{\mathsf{TM}}$ is undecidable

Alternate Proof: EQ_{TM} is undecidable

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Flashback: proof by contradiction:

- Assume EQ_{TM} has *decider* R; use to create E_{TM} *decider*:

$= \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

$S =$ "On input $\langle M\rangle$, where M is a TM:

1. Run R on input $\langle M, M_1\rangle$, where M_1 is a TM that rejects all inputs.
2. If R accepts, *accept*; if R rejects, *reject*."

Alternate proof: Show: $E_{\mathsf{TM}} \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$

- Computable fn f : $\langle M\rangle$ → $\langle M, M_1\rangle$

DEFINITION 5.20

Language A is ***mapping reducible*** to language B, written $A \leq_{\mathrm{m}} B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \iff f(w) \in B.$$

The function f is called the ***reduction*** from A to B.

Reducing to complement: E_{TM} is undecidable

$E_{TM} = \{\langle M \rangle |\ M \text{ is a TM and } L(M) = \emptyset\}$

Proof, by contradiction:

- Assume E_{TM} has *decider* R; use to create A_{TM} *decider*:

$S =$ "On input $\langle M, w \rangle$, an encoding of a TM M and a string w:

1. Use the description of M and w to construct the TM M_1 just described.

$M_1 =$ "On input x:
1. If $x \neq w$, *reject*.
2. If $x = w$, run M on input w and *accept* if M does."

2. Run R on input $\langle M_1 \rangle$.

3. If R accepts, *reject*; if R rejects, *accept*."

If M accepts w, M_1 not in E_{TM}!

Alternate proof: computable fn: $\langle M, w \rangle \rightarrow \langle M_1 \rangle$???

- So this only reduces A_{TM} to $\overline{E_{TM}}$
- Still proves E_{TM} is undecidable
  - HW9: show that undecidable langs are closed under complement

More Helpful Theorems

If $A \leq_m B$ and B is Turing-recognizable, then A is Turing-recognizable.

If $A \leq_m B$ and A is not Turing-recognizable, then B is not Turing-recognizable.

- Same proofs as:

If $A \leq_m B$ and B is decidable, then A is decidable.

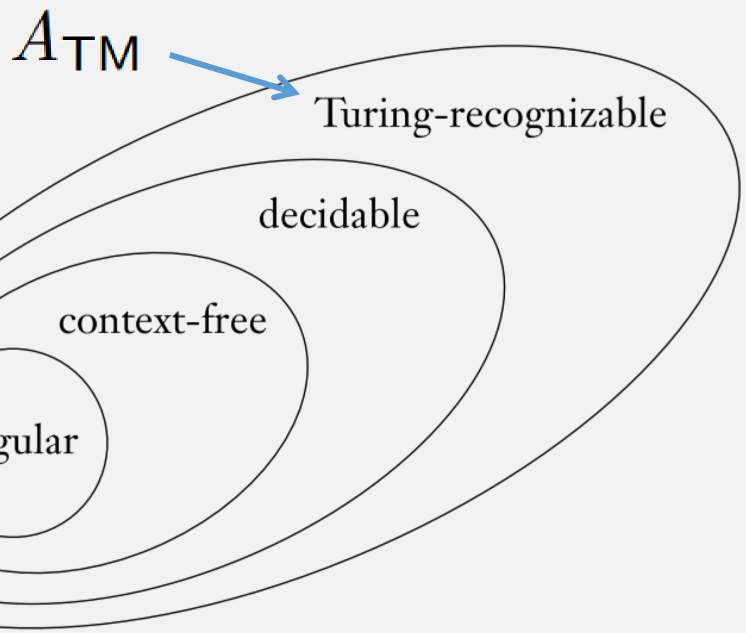
If $A \leq_m B$ and A is undecidable, then B is undecidable.

Thm: EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

1. EQ_{TM} is not Turing-recognizable

$\overline{A_{\mathsf{TM}}}$



$\overline{A_{\mathsf{TM}}} \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$ A is not Turing-recognizable, th EQ_{TM} not Turing-recognizable.

Mapping Reducibility implies Mapping Red. of Complements

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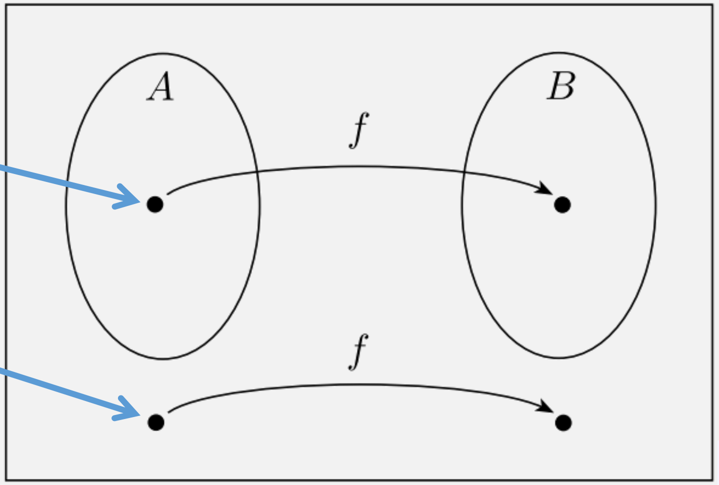
The function f is called the ***reduction*** from A to B.

$A \leq_m B$

implies

$\overline{A} \leq_m \overline{B}$



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A function $f\colon \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Thm: EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

1. EQ_{TM} is not Turing-recognizable

Two Choices:

- Create Computable fn: $\overline{A_{\mathsf{TM}}} \rightarrow EQ_{\mathsf{TM}}$
- Or Computable fn: $A_{\mathsf{TM}} \rightarrow \overline{EQ_{\mathsf{TM}}}$

Thm: EQ_{TM} is not Turing-recognizable

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle |\ M_1$ and M_2 are TMs and $L(M_1) = L(M_2)\}$

- Create Computable fn: $A_{\mathsf{TM}} \rightarrow \overline{EQ_{\mathsf{TM}}}$
- $\langle M, w\rangle \rightarrow \langle M_1, M_2\rangle$ M_1 and M_2 are TMs and $L(M_1) \neq L(M_2)$

$F =$ "On input $\langle M, w\rangle$, where M is a TM and w a string:

1. Construct the following two machines, M_1 and M_2.

   $M_1 =$ "On any input: ← Accepts nothing

   1. *Reject*."

   $M_2 =$ "On any input: ← Accepts nothing or everything

   1. Run M on w. If it accepts, *accept*."

2. Output $\langle M_1, M_2\rangle$."

- If M accepts w, M_1 not equal to M_2
- If M does not accept w, M_1 equal to M_2

Thm: EQ_{TM} is neither Turing-recognizable nor co-Turing-recognizable.

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

1. EQ_{TM} is not Turing-recognizable

- Create Computable fn: $\overline{A_{\mathsf{TM}}} \rightarrow EQ_{\mathsf{TM}}$
- Or Computable fn: $A_{\mathsf{TM}} \rightarrow \overline{EQ_{\mathsf{TM}}}$
- **DONE!**

2. $\overline{EQ_{\mathsf{TM}}}$ is not ~~co~~-Turing-recognizable

- (A lang is co-Turing-recog. if it is complement of Turing-recog. lang)

<u>Prev</u>: EQ_{TM} is not Turing-recognizable

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1$ and M_2 are TMs and $L(M_1) = L(M_2)\}$

- Create <u>Computable fn</u>: A_{TM} → $\overline{EQ_{\mathsf{TM}}}$
- $\langle M, w\rangle$ → $\langle M_1, M_2\rangle$ M_1 and M_2 are TMs and $L(M_1) \neq L(M_2)$

$F =$ "On input $\langle M, w\rangle$, where M is a TM and w a string:

1. Construct the following two machines, M_1 and M_2.

$M_1 =$ "On any input: ← Accepts nothing

1. *Reject*."

$M_2 =$ "On any input: ← Accepts nothing or everything

1. Run M on w. If it accepts, *accept*."

2. Output $\langle M_1, M_2\rangle$."

DONE!

Now: $\overline{EQ}_{\text{TM}}$ is not Turing-recognizable

$EQ_{\text{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

- Create Computable fn: $A_{\text{TM}} \rightarrow \overline{EQ}_{\text{TM}}$
- $\langle M, w\rangle \rightarrow \langle M_1, M_2\rangle$ M_1 and M_2 are TMs and $L(M_1) \neq L(M_2)$

$F =$ "On input $\langle M, w\rangle$, where M is a TM and w a string:

1. Construct the following two machines, M_1 and M_2.

   $M_1 =$ "On any input: ← Accepts ~~nothing~~ everything

   1. *Accept*."

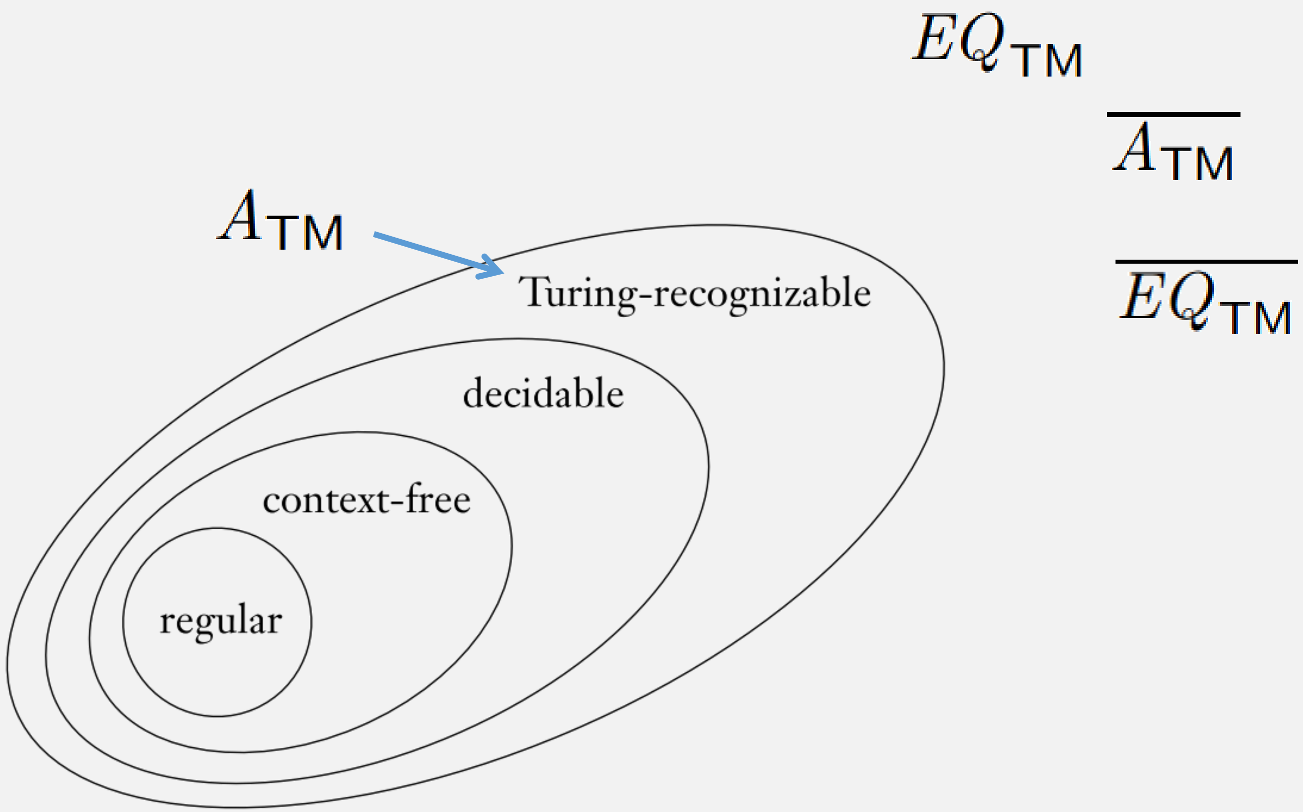
   $M_2 =$ "On any input: ← Accepts nothing or everything

   1. Run M on w. If it accepts, *accept*."

2. Output $\langle M_1, M_2\rangle$."

DONE!

Unrecognizable Languages



Check-in Quiz 4/7

On gradescope