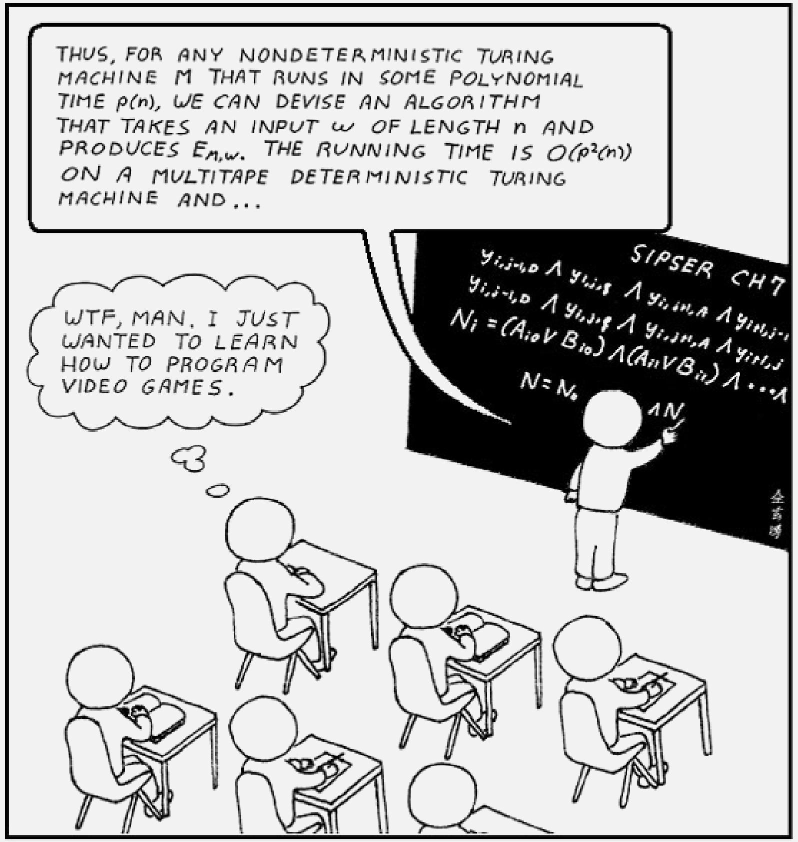
CS420, Ch7
Time Complexity

Wed April 14, 2021

Announcements

- Reminder: No class next Monday 4/19!
- HW9
  - Due date extended to Tues 4/20 11:59pm EST
- HW10 coming soon!
- FAQ: How many HWs left?
  - Total: 12 HWs
- FAQ: What's my grade?
  - All your scores are visible in Gradescope
  - Letter grade brackets: 90s -> A, 80s, -> B, etc.
    - See: CS420 Spring 2021 Course Page -> Course Policies -> Grading
  - Final grade, incl. dropped hw, particip, not calculated until end of semester

Flashback: Single-tape TM “equiv to“ Nondet. TM



Flashback: Single-tape TM "equiv to" Nondet. TM

- Deterministic TM simulating nondeterministic TM:
  1. Number the nodes at each step
  2. Deterministically check every path, in breadth-first order (restart at top each time)
     - 1
     - 1-1
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     - 1-1-1
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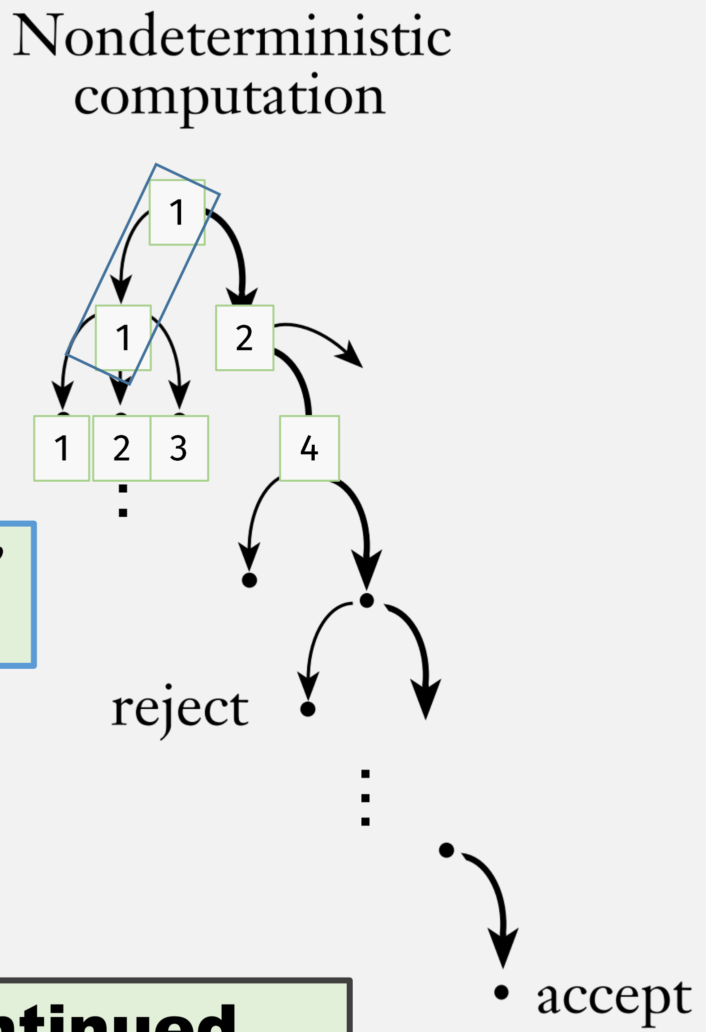
     - and so on
  3. Accept if accepting config found

"This is the most inefficient algorithm ever"
--- CS420 Spring 2021 student

Exactly how inefficient is it???

Now we'll start to count "# of steps"

To be continued …



Simpler Example: $A = \{0^k 1^k | \, k \geq 0\}$

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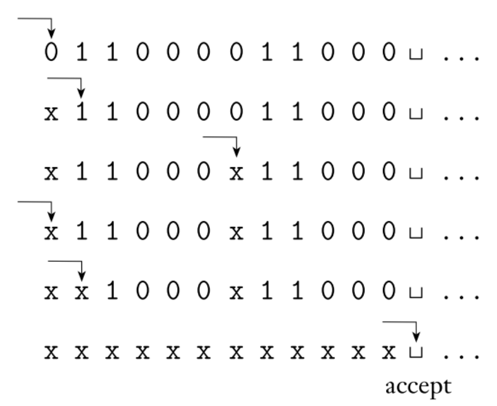

Number of steps (worst case), **n = length of input:**

➢ TM Line 1:

- **n steps to scan + n steps to return to beginning = 2n steps**

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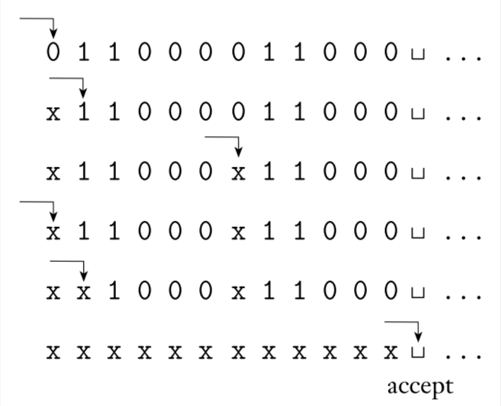

Number of steps (worst case), **n** = **length of input:**

- TM Line 1:
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➢ Lines 2 and 3 (loop):
  - Each iter: **n/2** steps to find "**1**" + **n/2** steps to return = **n** steps
  - Num iters: Each scan crosses off 2 chars, so at most **n/2** scans
  - Total = each iter times num iters = **n (n/2)** = **$n^2/2$ steps**

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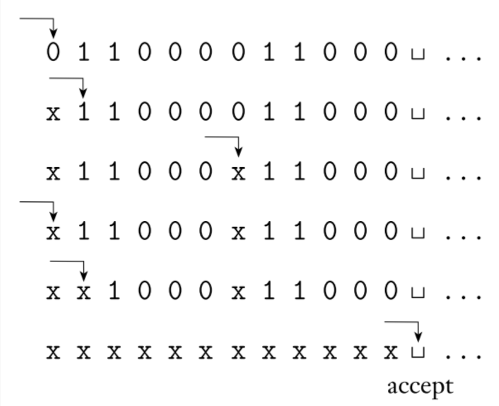
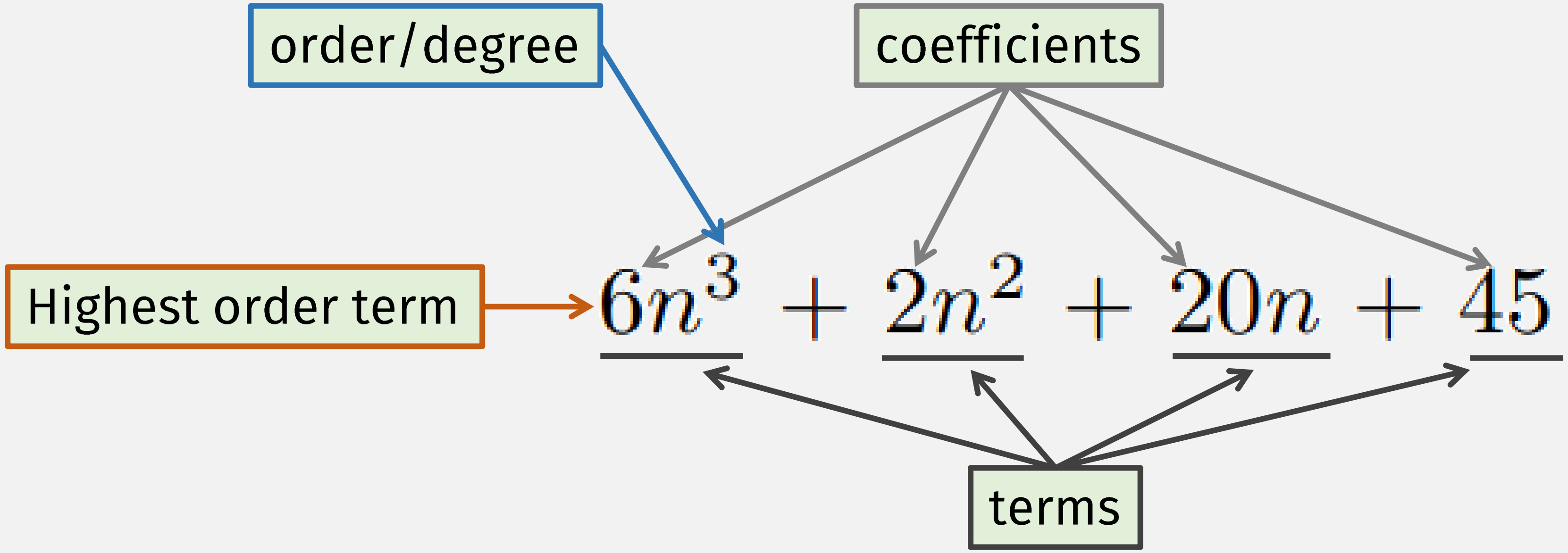


Number of steps (worst case), **n = length of input:**

- TM Line 1:
  - **n steps to scan + n steps to return to beginning = 2n steps**
- Lines 2 and 3 (loop):
  - Each iter: **n/2** steps to find "1" + **n/2** steps to return = **n** steps
  - Num iters: Each scan crosses off 2 chars, so at most **n/2** scans
  - Total = each iter times num iters = **n (n/2)** = **$n^2/2$ steps**
- ➢ Line 4:
  - **n steps** to scan input one more time
- Total: **$2n + n^2/2 + n$ = $n^2/2 + 3n$ steps**

Interlude: Polynomials



Definition: Time Complexity

NOTE: n has no units, it's only *roughly* "length" of the input

But n can be not only #characters, but also #states, #nodes, etc.

We can use any of things for n, bc they're correlated with input length

DEFINITION 7.1

et M be a deterministic Turing machine that halts on all in-
uts. The ***running time*** or ***time complexity*** of M is the function
: $\mathcal{N} \longrightarrow \mathcal{N}$, where $f(n)$ is the maximum number of steps that M
uses on any input of length n. If $f(n)$ is the running time of M,
say that M runs in time $f(n)$ and that M is an $f(n)$ time Tur-
machine. Customarily we use n to represent the length of the
ut.

- Machine M_1 that decides $A = \{0^k 1^k | k \geq 0\}$
  - Running Time: $\mathbf{n^2/2+3n}$

M_1 = "On input string w:
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Interlude: Asymptotic Analysis

- Total: $\mathbf{n^2 + 3n}$
  - If $\mathbf{n} = 1$
    - $\mathbf{n^2} = 1$
    - $\mathbf{3n} = 3$
    - Total $= 4$
  - If $\mathbf{n} = 10$
    - $\mathbf{n^2} = 100$
    - $\mathbf{3n} = 30$
    - Total $= 130$
  - If $\mathbf{n} = 100$
    - $\mathbf{n^2} = 10000$
    - $\mathbf{3n} = 300$
    - Total $= 10300$
  - If $\mathbf{n} = 1000$
    - $\mathbf{n^2} = 1000000$
    - $\mathbf{3n} = 3000$
    - Total $= 1003000$
- $\mathbf{n^2 + 3n} \approx \mathbf{n^2}$ as n gets large
- asymptotic analysis only cares about large n

Definition: Big-O Notation

DEFINITION 7.2

Let f and g be functions $f, g \colon \mathcal{N} \longrightarrow \mathcal{R}^+$. Say that $\boldsymbol{f(n) = O(g(n))}$ if positive integers c and n_0 exist such that for every integer $n \geq n_0$,

$$f(n) \leq c\, g(n).$$

When $f(n) = O(g(n))$, we say that $g(n)$ is an ***upper bound*** for $f(n)$, or more precisely, that $g(n)$ is an ***asymptotic upper bound*** for $f(n)$, to emphasize that we are suppressing constant factors.

- In English: Keep only highest order term, drop all coefficients
- Machine M_1 that decides $A = \{0^k 1^k \mid k \geq 0\}$
  - Is an $\mathbf{n^2 + 3n}$ time Turing machine
  - Is an $O(\mathbf{n^2})$ time Turing machine
  - Has asymptotic upper bound $O(\mathbf{n^2})$

Definition: Small-o Notation (less used)

DEFINITION 7.5

Let f and g be functions $f, g \colon \mathcal{N} \longrightarrow \mathcal{R}^+$. Say that $\boldsymbol{f(n) = o(g(n))}$ if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$

In other words, $f(n) = o(g(n))$ means that for any real number $c > 0$, a number n_0 exists, where $f(n) < c\,g(n)$ for all $n \geq n_0$.

- Analogy:
  - Big-O : <= :: small-o : <

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Big-O arithmetic

- $O(\mathbf{n^2}) + O(\mathbf{n^2})$
  $= O(\mathbf{n^2})$

- $O(\mathbf{n^2}) + O(\mathbf{n})$
  $= O(\mathbf{n^2})$

Definition: Time Complexity Classes

DEFINITION 7.7

Let $t\colon \mathcal{N} \longrightarrow \mathcal{R}^+$ be a function. Define the ***time complexity class***, $\mathbf{TIME}(t(n))$, to be the collection of all languages that are decidable by an $O(t(n))$ time Turing machine.

TMs have a running time, languages have a complexity class

- Machine M_1 that decides $A = \{0^k 1^k \mid k \geq 0\}$
  - Is an $O(\mathbf{n}^2)$ running time Turing machine
  - So A is in $\mathrm{TIME}(\mathbf{n}^2)$

A Faster Machine? $A = \{0^k 1^k | \, k \geq 0\}$

$M_2 =$ "On input string w:

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2. Repeat as long as some 0s and some 1s remain on the tape:
3. Scan across the tape, checking whether the total number of 0s and 1s remaining is even or odd. If it is odd, *reject*.
4. Scan again across the tape, crossing off every other 0 starting with the first 0, and then crossing off every other 1 starting with the first 1.
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- ➢ Lines 2, 3, 4 (loop):
  - Each iter: a scan takes *O*(**n**) steps
  - Num iters: Each iter crosses off *half* the chars, so at most *O*(**log n**) scans
  - Total: *O*(**n**) * *O*(**log n**) = *O*(**n log n**) steps

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Number of steps (worst case), **n = length of input:**

- Line 1:
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  - Total: $O(\mathbf{n}) * O(\mathbf{log\ n}) = O(\mathbf{n\ log\ n})$ steps
- Line 5:
  - $O(\mathbf{n})$ steps to scan input one more time
- Total: $O(\mathbf{n}) + O(\mathbf{n\ log\ n}) + O(\mathbf{n}) = O(\mathbf{n\ log\ n})$ steps

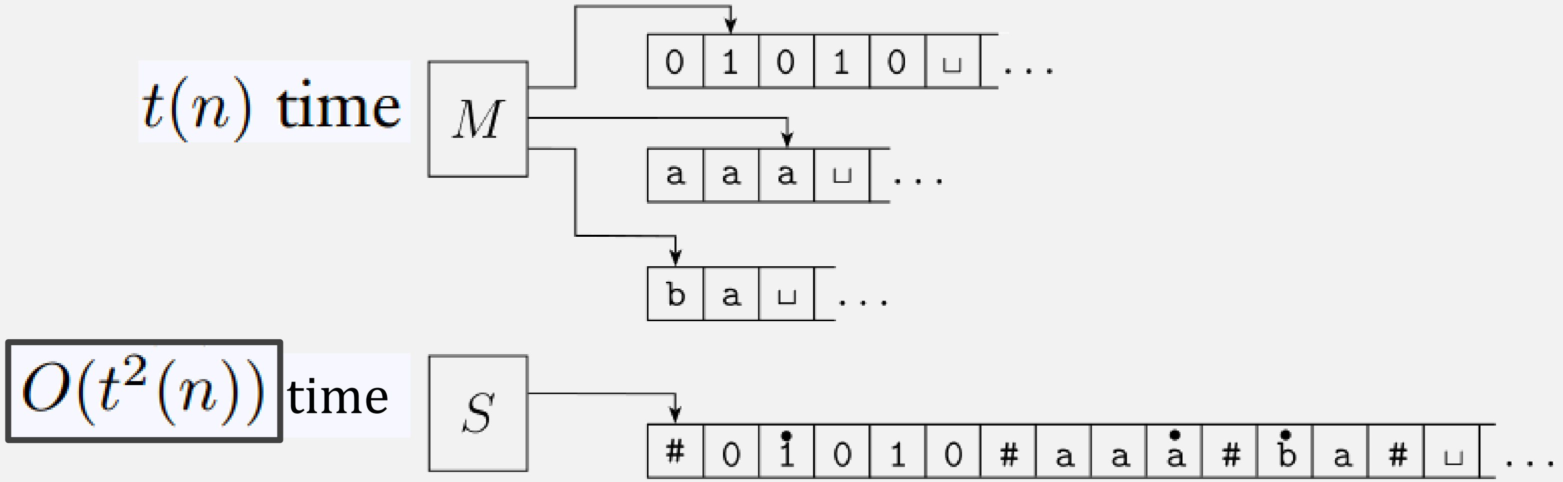
Interlude: Logarithms

- $2^x = y$
- $\log_2 y = x$
- $\log_2 n = O(\mathbf{log\ n})$
  - "divide and conquer" algorithms = $O(\mathbf{log\ n})$
  - E.g., binary search
- In computer science, **base-2 is the only base!**

Terminology: Categories of Bounds

- Exponential time
  - $O(\mathbf{2^{n^c}})$, for $c > 0$ (always base 2)
- Polynomial time
  - $O(\mathbf{n^c})$, for $c > 0$
- Quadratic time (special case of polynomial time)
  - $O(\mathbf{n^2})$
- Linear time (special case of polynomial time)
  - $O(\mathbf{n})$
- Log time
  - $O(\mathbf{\log n})$

Multi-tape vs Single-tape TMs: # of Steps



- For single-tape TM to simulate 1 step of multi-tape:
  - Scan to find all “heads” = O(length of all M’s tapes)
  - “Execute” transition at all the heads = O(length of all M’s tapes)
- # single-tape steps to simulate 1 multitape step (worst case)
  - = O(length of all M’s tapes)
  - = $O(t(n))$ (If M spends all its steps expanding its tapes)
- <u>Total steps (single tape)</u>: $O(t(n))$ per step × $t(n)$ steps = $\boldsymbol{O(t^2(n))}$

Single-tape TM vs Nondet. TM: # of steps

- Deterministic TM simulating nondeterministic TM:
  - Number the nodes at each step
  - Deterministically check every path, in breadth-first order (restart at top each time)
    - 1
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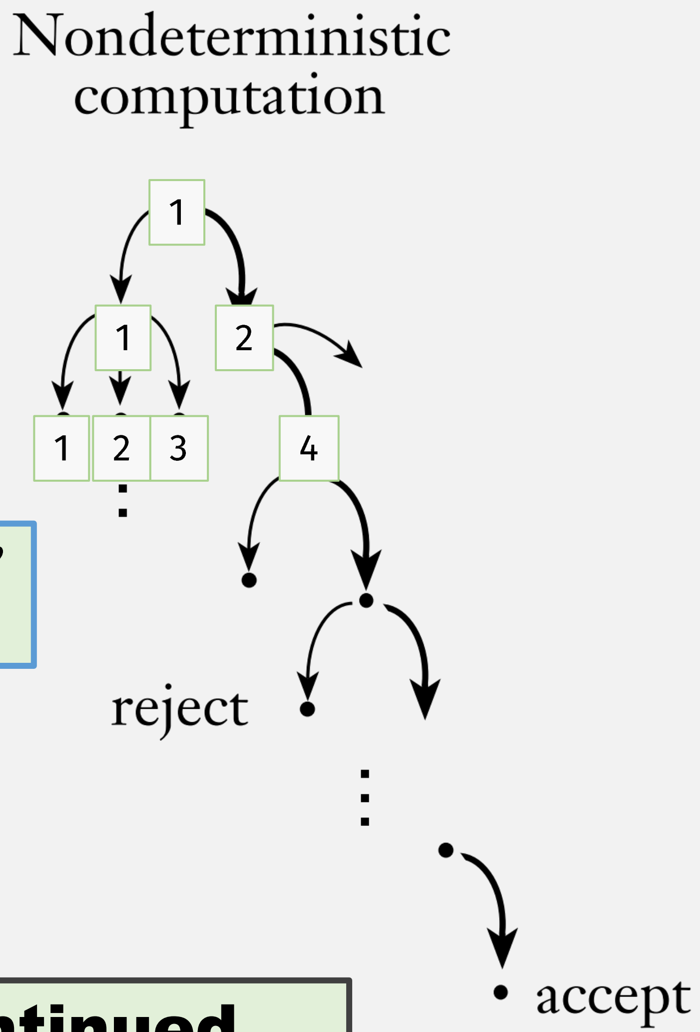
Exactly how inefficient is it???

Now we'll start to count "# of steps"



To be continued ...

Single-tape TM vs Nondet. TM: # of steps

$2^{O(t(n))}$ time

- Deterministic TM simulating nondeterministic TM: ← $t(n)$ time
  - Number the nodes at each step
  - Deterministically check every path, in breadth-first order (restart at top each time)
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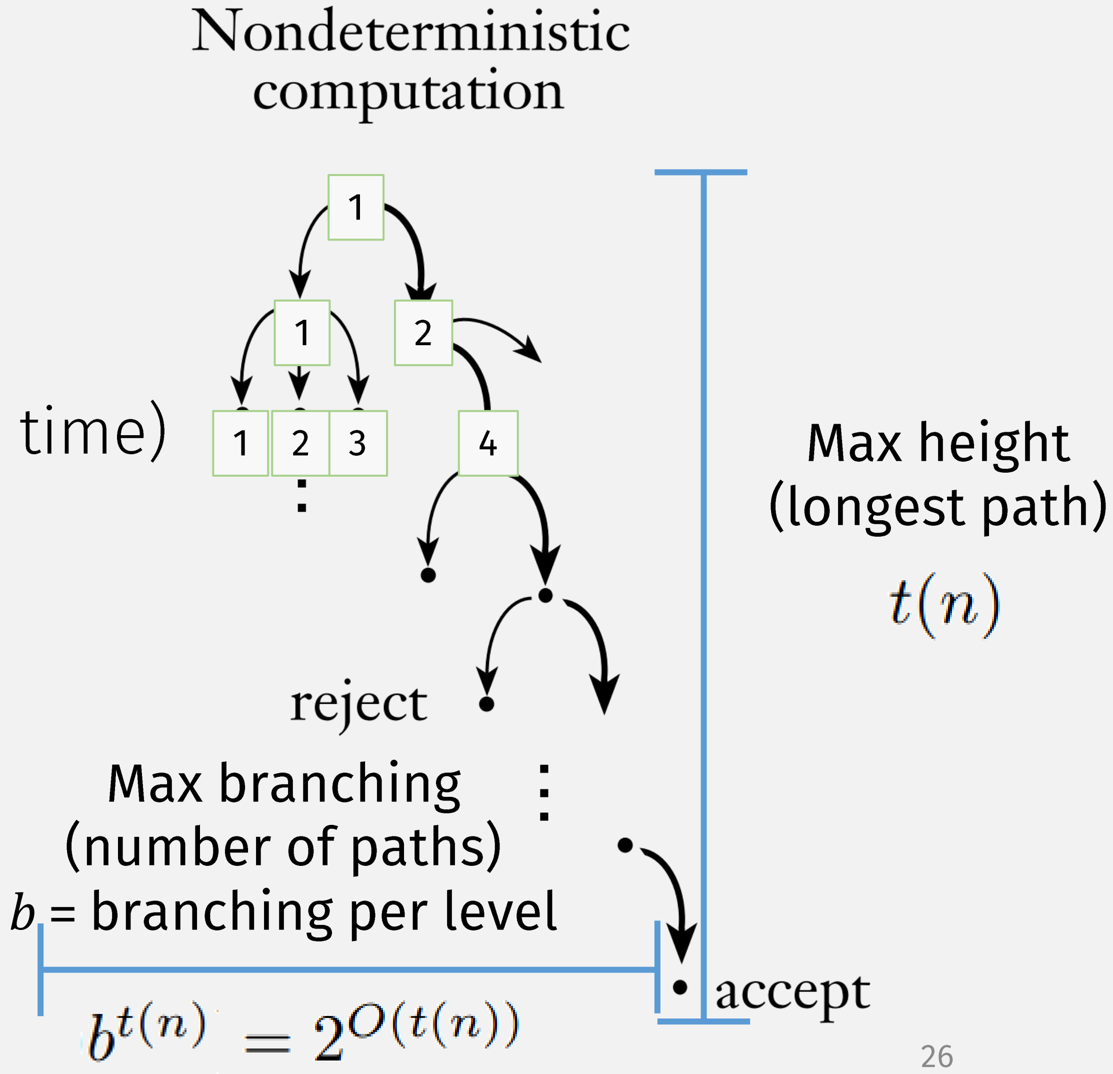


Summary

- If multi-tape TM: $t(n)$ time
- Then equivalent single-tape TM: $O(t^2(n))$
  - Quadratically slower

- If non-deterministic TM: $t(n)$ time
- Then equivalent single-tape TM: $2^{O(t(n))}$
  - Exponentially slower

Next time: Specific Complexity Classes

DEFINITION 7.12

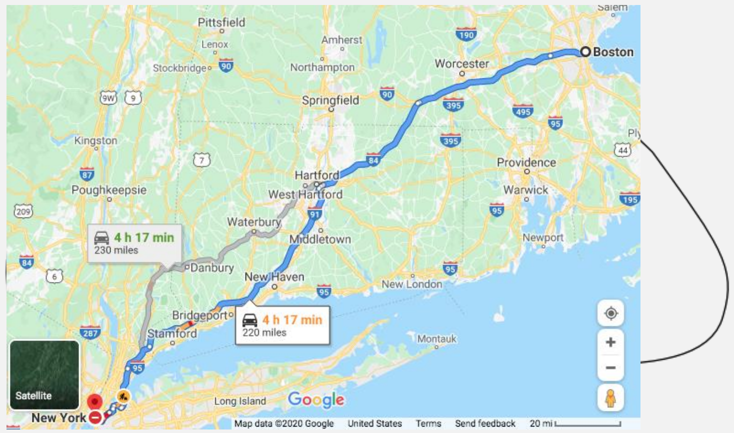
P is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$P = \bigcup_k \text{TIME}(n^k).$$

- Corresponds to "realistically" solvable problems
- In this class:
  - Problems in P = "solvable"
  - Problems outside P = "unsolvable"
    - These are usually "brute force" solutions that "try all possible inputs"

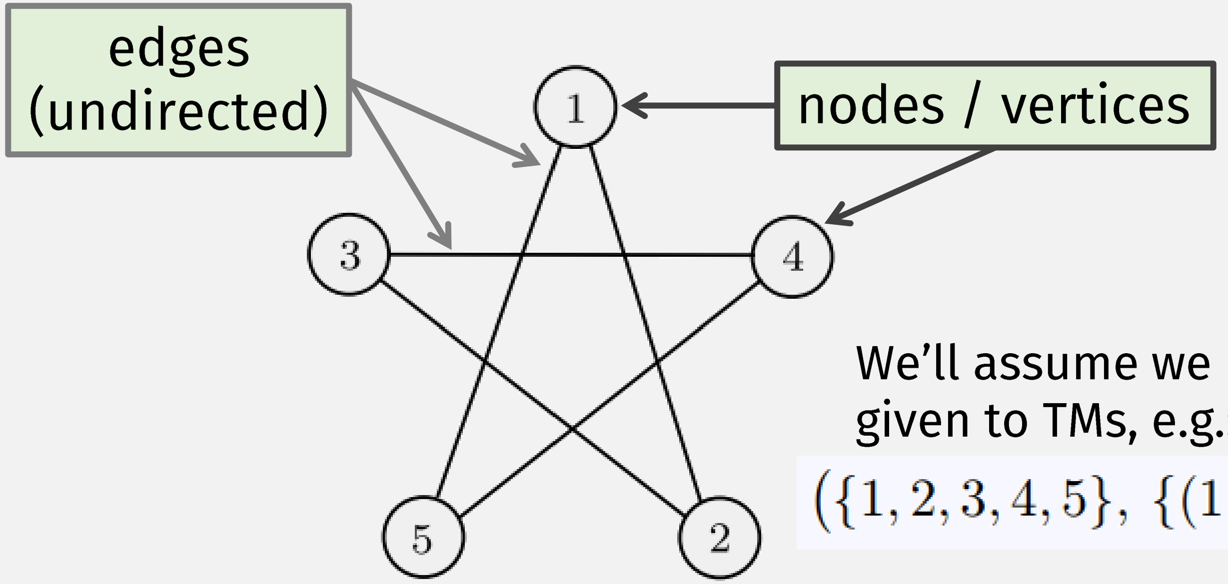
Next time: A Graph Theorem: $PATH \in \mathrm{P}$

$PATH = \{\langle G, s, t\rangle \mid G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$



- To prove that a language is in **P** …
- … we must construct a polynomial time algorithm deciding the lang
- A non-polynomial (i.e., exponential, brute force) algorithm:
  - Check all possible paths, and see if any connect s to t

Interlude: Graphs (see Chapter 0)



We'll assume we have some **string encoding**, <G>, given to TMs, e.g.:

$$(\{1, 2, 3, 4, 5\},\ \{(1, 2),\ (2, 3),\ (3, 4),\ (4, 5),\ (5, 1)\})$$

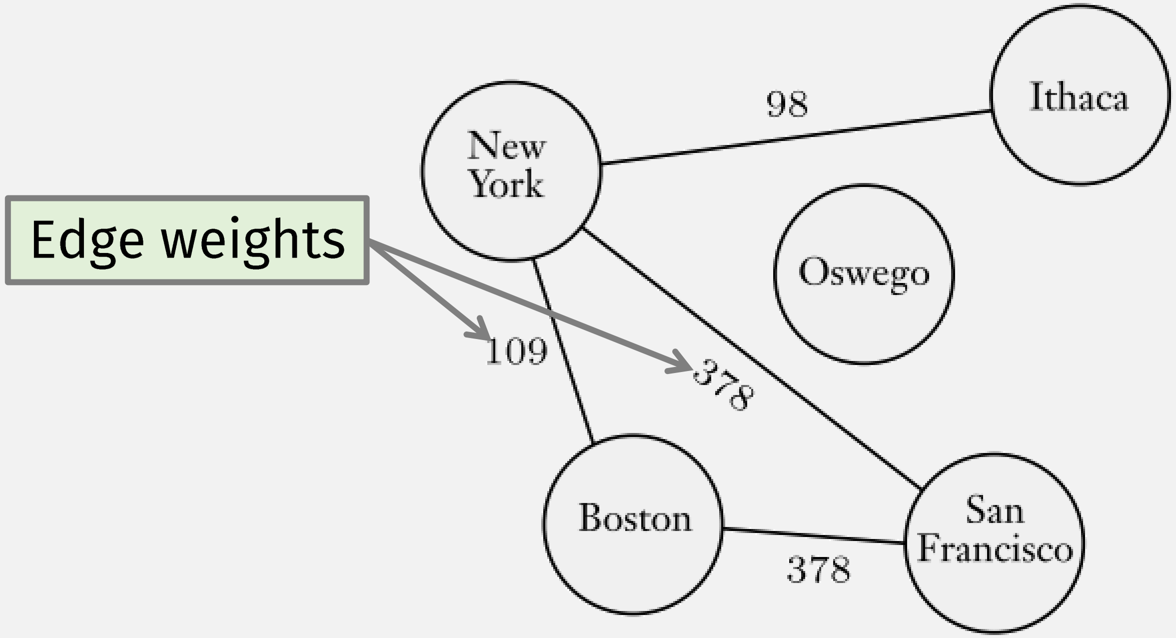
- Edge defined by two nodes (order doesn't matter)
- Formally, a graph = *(V, E)*
  - V = set of nodes, E = set of edges

Interlude: Graph Encodings

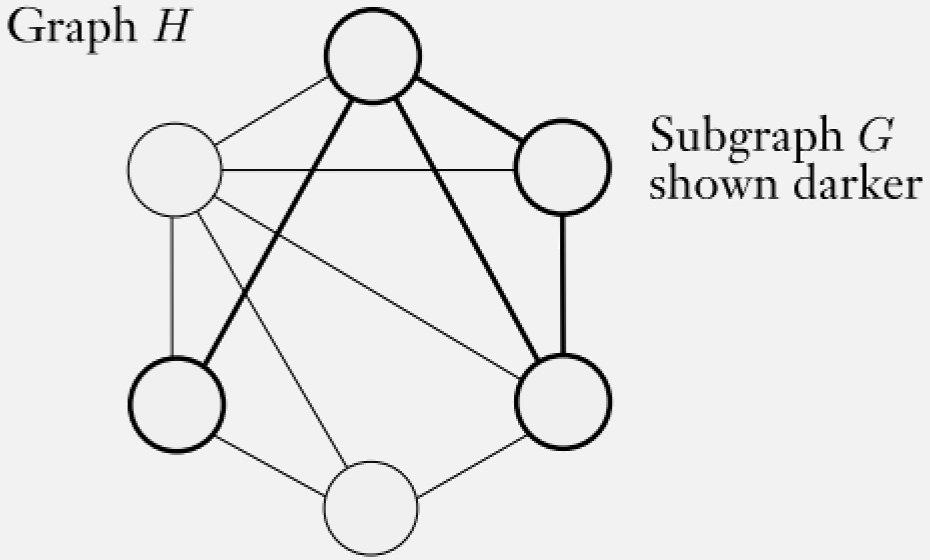
$(\{1, 2, 3, 4, 5\}, \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 1)\})$

- In graph algorithms, “length of input” n = number of vertices
  - and sometimes number of edges
  - Not number of chars
  - So steps counted in terms of number of vertices
- Given a graph $G = (V, E)$ with $\mathrm{n} = |V|$ vertices
- Max edges = $O(|\boldsymbol{V}|^2) = O(\mathbf{n}^2)$
- So # vertices + edges is polynomial in length of input
- Algorithm runs in time polynomial in the number of vertices ⇔ algorithm runs in time polynomial in the length of input

Interlude: Weighted Graphs

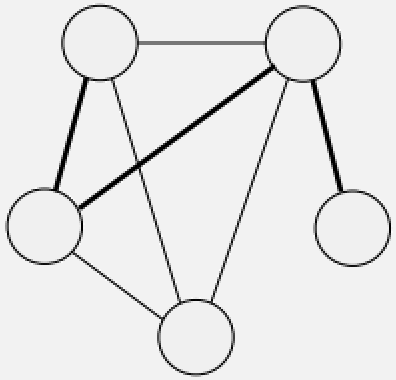


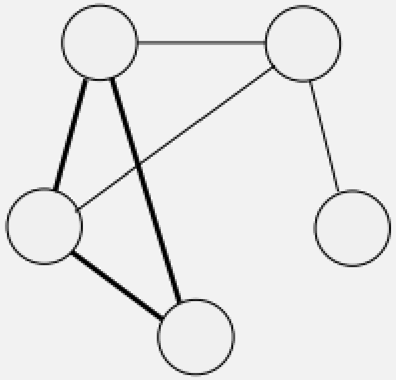
Interlude: Subgraphs

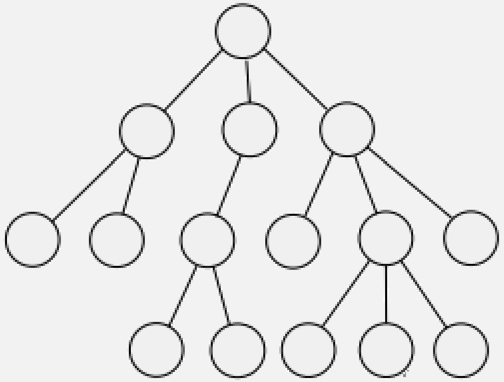


Interlude: Paths and other Graph Things

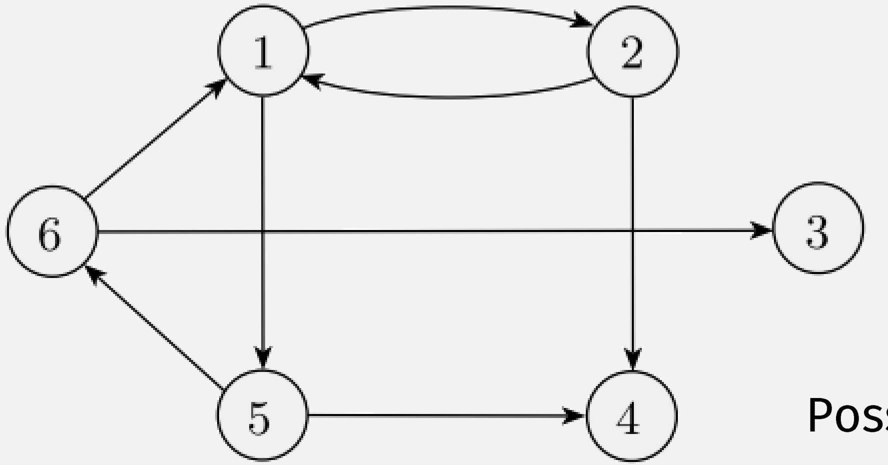
- Path
  - A sequence of nodes connected by edges
- Cycle
  - A path that starts/ends at the same node
- Connected graph
  - Every two nodes has a path
- Tree
  - A connected graph with no cycles







Interlude: Directed Graphs



Possible **string encoding** given to TMs:

$(\{1,2,3,4,5,6\}, \{(1,2), (1,5), (2,1), (2,4), (5,4), (5,6), (6,1), (6,3)\})$

Each pair of nodes included twice

- Directed graph = *(V, E)*
  - V = set of nodes, E = set of edges
- An edge is a pair of nodes *(u,v)*, order now matters
  - u = “from” node, v = “to” node
- A “degree” of a node is the number of edges connected to the node
  - Nodes in a directed graph have both indegree and outdegree

Check-in Quiz 4/14

On gradescope