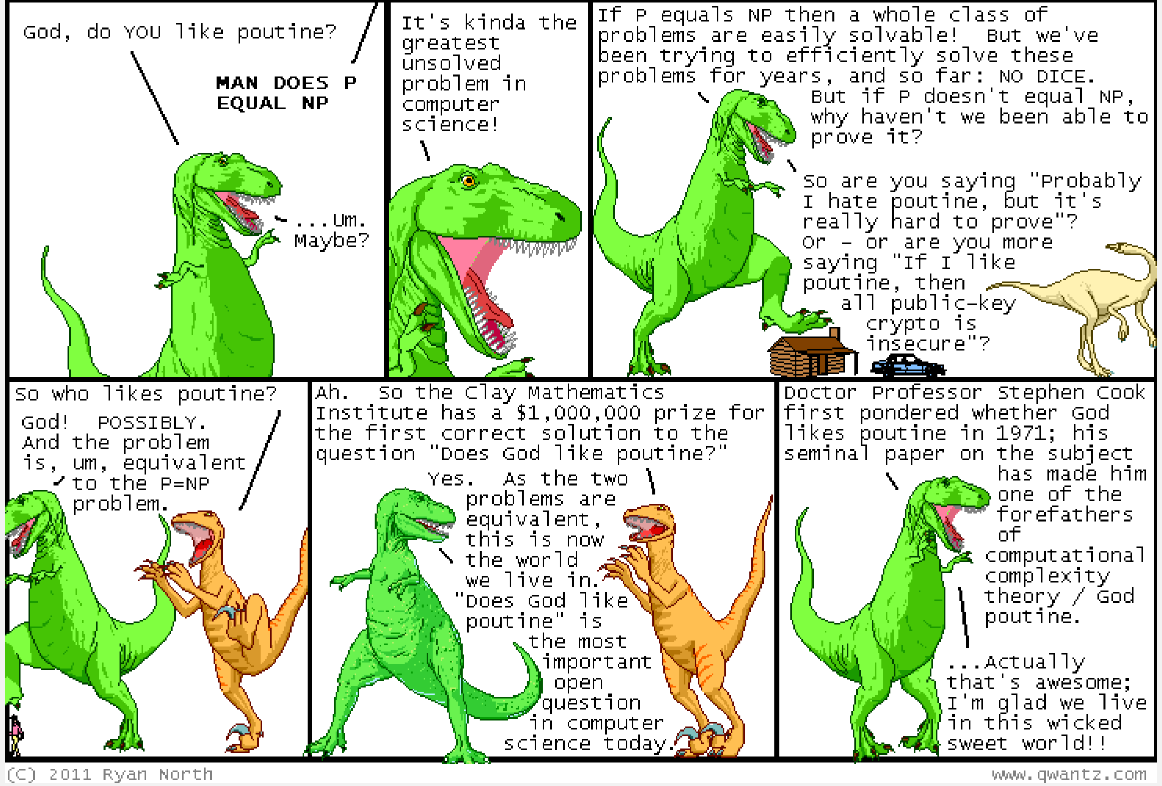
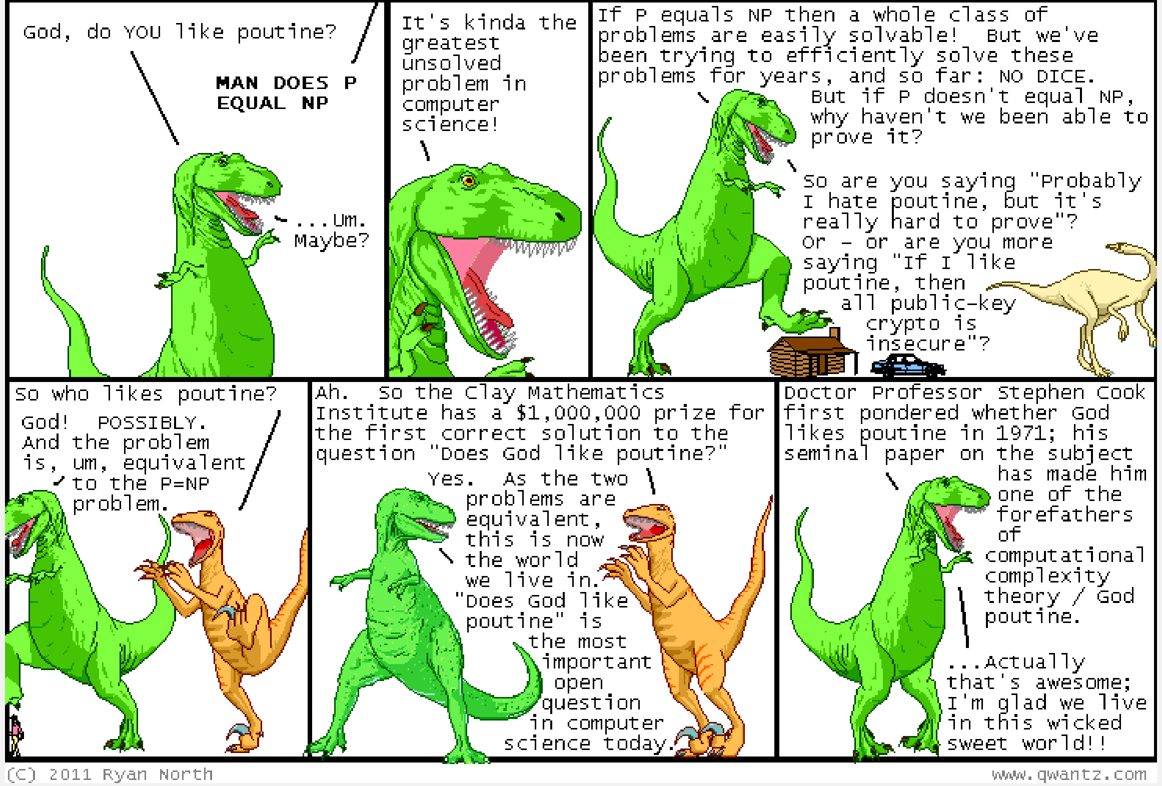


Poly Time Mapping Reducibility

Wednesday, April 28, 2021

Announcements

- HW 10 past due
- HW 11 released
  - Due Tues 5/4 11:59pm EST



Last Time: **P** vs **NP**

- **P** = class of languages that can be decided "quickly"
  - i.e., "solvable" with a deterministic TM

- **NP** = class of languages that can be verified "quickly"
  - or, "solvable" with a nondeterministic TM

- Does **P** = **NP** ?
  - Problem first posed by John Nash

- It's a difficult problem because how do you prove:
  "we'll never find a poly time algorithm for X"?

Progress on whether **P = NP** ?

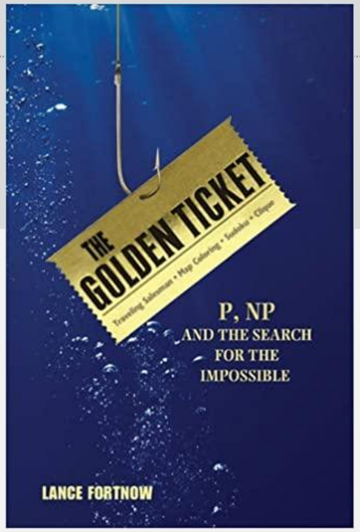
- Still not close

$P \stackrel{?}{=} NP$

Scott Aaronson*

The Status of the P Versus NP Problem

By Lance Fortnow
Communications of the ACM, September 2009, Vol. 52 No. 9, Pages 78-86
10.1145/1562164.1562186



- One important concept discovered:
  - **NP**-Completeness (today)

Flashback: Mapping Reducibility

DEFINITION 5.20

Language A is ***mapping reducible*** to language B, written $A \leq_m B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

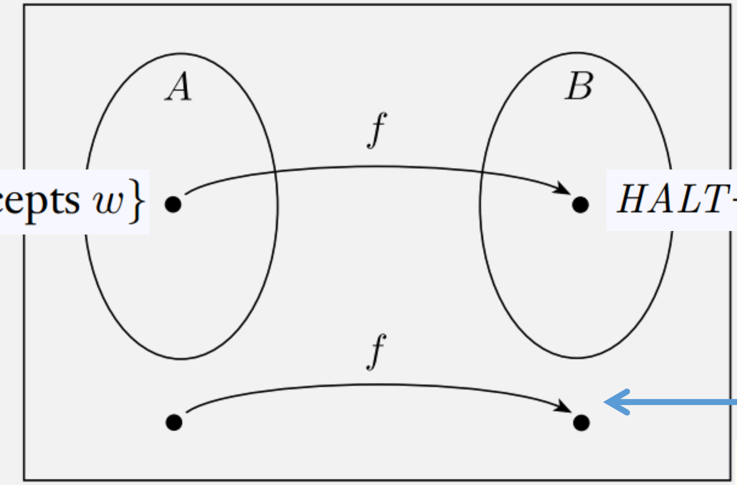
$$w \in A \iff f(w) \in B.$$

The function f is called the ***reduction*** from A to B.

IMPORTANT: "if and only if" ...

So to show mapping reducibility:
1. must create **computable fn**
2. and then show **forward direction**
3. and **reverse direction**

$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$



$HALT_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$

... means $\overline{A} \leq_m \overline{B}$

DEFINITION 5.17

A function $f\colon \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Polynomial Time Mapping Reducibility

DEFINITION 5.20

Language A is ***mapping reducible*** to language B, written $A \leq_m B$, if there is a computable function $f: \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \iff f(w) \in B.$$

The function f is called the ***reduction*** from A to B.

DEFINITION 7.29

Language A is ***polynomial time mapping reducible***,[1] or simply ***polynomial time reducible***, to language B, written $A \leq_P B$, if a polynomial time computable function $f: \Sigma^* \longrightarrow \Sigma^*$ exists, where for every w,

$$w \in A \iff f(w) \in B.$$

The function f is called the ***polynomial time reduction*** of A to B.

Don't Forget: "if and only if" …

DEFINITION 5.17

A function $f: \Sigma^* \longrightarrow \Sigma^*$ is a *poly time* ***computable function*** if some *poly time* Turing machine M, on every input w, halts with just $f(w)$ on its tape.

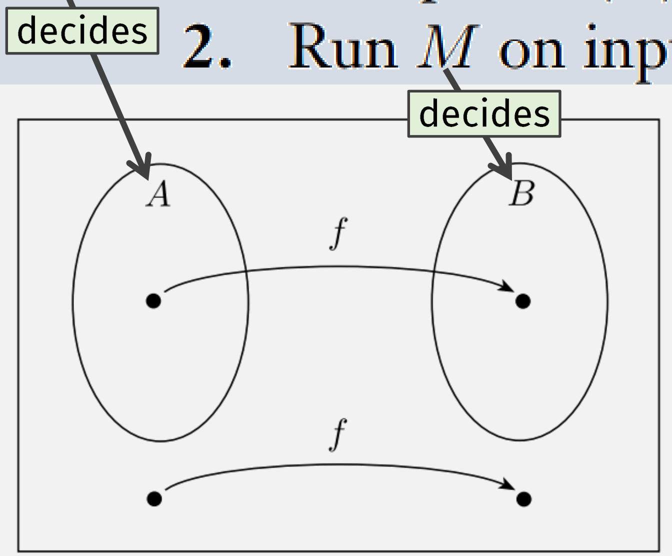
Flashback: If $A \leq_m B$ and B is decidable, then A is decidable.

(Theorem 5.22)

PROOF We let M be the decider for B and f be the reduction from A to B. We describe a decider N for A as follows.

N = "On input w:

1. Compute $f(w)$.
2. Run M on input $f(w)$ and output whatever M outputs."



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THEOREM 7.31 If $A \leq$~~$_{\text{m}}$~~$_{\mathbf{P}}$ B and B ~~is decidable~~ $\in$ P, then A ~~is decidable~~ $\in$ P.

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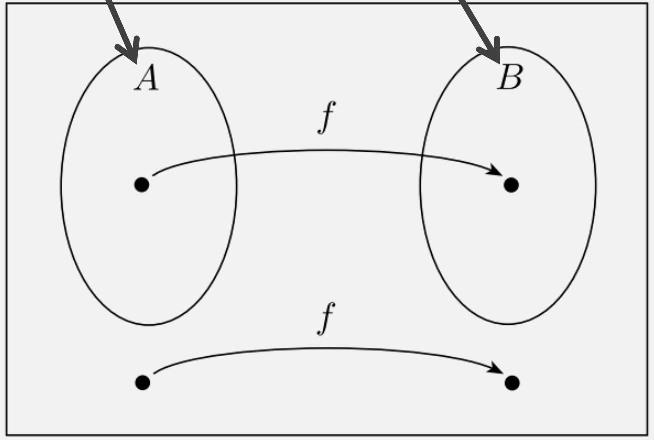

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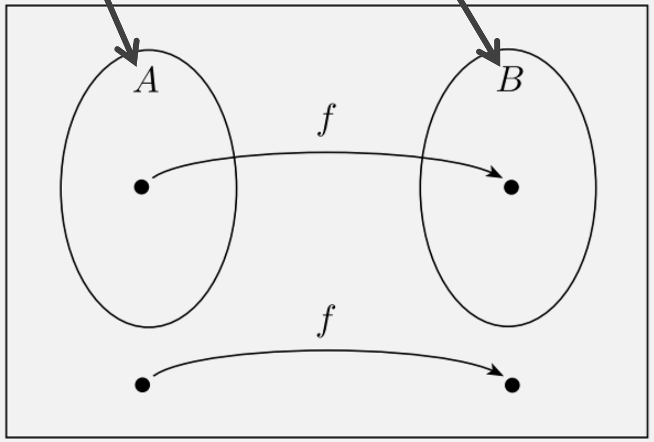


DEFINITION 5.20

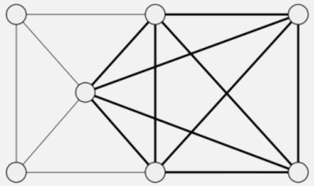
Language A is poly time *mapping reducible* to language B, written $A \leq_m B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \Longleftrightarrow f(w) \in B.$$

The function f is called the ***reduction*** from A to B.

Theorem: *3SAT* is polynomial time reducible to *CLIQUE*.

Last Class: *CLIQUE* is in NP



$CLIQUE = \{\langle G, k\rangle \mid G$ is an undirected graph with a k-clique$\}$

PROOF IDEA The clique is the certificate.

PROOF The following is a verifier V for *CLIQUE*.

V = "On input $\langle\langle G, k\rangle, c\rangle$:
1. Test whether c is a subgraph with k nodes in G. $O(k)$
2. Test whether G contains all edges connecting nodes in c. $O(k^2)$
3. If both pass, *accept*; otherwise, *reject*."

DEFINITION 7.18

A ***verifier*** for a language A is an algorithm V, where

$$A = \{w \mid V \text{ accepts } \langle w, c\rangle \text{ for some string } c\}.$$

We measure the time of a verifier only in terms of the length of w, so a ***polynomial time verifier*** runs in polynomial time in the length of w. A language A is ***polynomially verifiable*** if it has a polynomial time verifier.

DEFINITION 7.19

NP is the class of languages that have polynomial time verifiers.

Theorem: *3SAT* is polynomial time reducible to *CLIQUE*.

??

Boolean Formulas

A Boolean _____	Is …	Example:
Value	TRUE or FALSE (or 1 or 0)	TRUE, FALSE

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Formula ϕ	Combines **vars** and **operations**	$(\overline{x} \wedge y) \vee (x \wedge \overline{z})$

Boolean Satisfiability

- A Boolean formula is <u>satisfiable</u> if …

- … there is some assignment of TRUE or FALSE (1 or 0) to its variables that makes the entire formula TRUE

- Is $(\overline{x} \wedge y) \vee (x \wedge \overline{z})$ satisfiable?
  - Yes
  - $x = 0, y = 1, z = 0$

The Boolean Satisfiability Problem

$SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable Boolean formula}\}$

Show *SAT* is in **NP**:

- Let $\boldsymbol{n}$ = the number of variables in the formula

- Verifier:
  - Let the certificate c be some assignment of variables to values
  - Verifying whether this assignment satisfies the formula takes time $O(\boldsymbol{n})$

- Non-deterministic Decider:
  - Non-deterministically try all possible assignments in parallel
  - Checking each assignment again takes time $O(\boldsymbol{n})$

- What about *3SAT*?

More Boolean Formulas

A Boolean _____	Is …	Example:
Value	TRUE or FALSE (or 1 or 0)	TRUE, FALSE
Variable	Represents a Boolean value	x, y, z
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Literal	A var or a negated var	x or $\overline{x}$.

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Conjunctive Normal Form (**CNF**)	**Clauses** ANDed together	$(x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4) \wedge (x_3 \vee \overline{x_5} \vee x_6)$

More Boolean Formulas

A Boolean _____	Is ...	Example:
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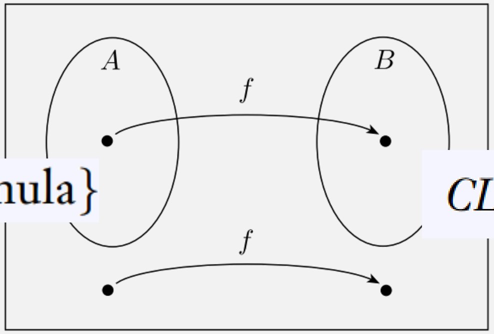
Conjunctive Normal Form (**CNF**)	**Clauses** ANDed together	$(x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4) \wedge (x_3 \vee \overline{x_5} \vee x_6)$
3CNF Formula	Three **literals** in each **clause**	$(x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge (x_3 \vee \overline{x_5} \vee x_6) \wedge (x_3 \vee \overline{x_6} \vee x_4)$

The *3SAT* Problem

$$3SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable 3cnf-formula}\}$$

Theorem: *3SAT* is polynomial time reducible to *CLIQUE*

$3SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable 3cnf-formula}\}$



$CLIQUE = \{\langle G, k\rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$

Need: poly time computable fn converting a 3cnf-formula …

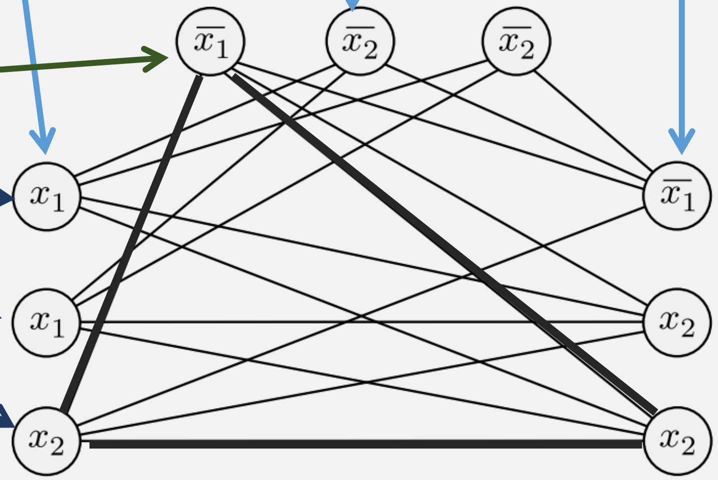
$$\phi = (x_1 \vee x_1 \vee x_2) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_2}) \wedge (\overline{x_1} \vee x_2 \vee x_2)$$

- … to a graph containing a clique:
  - Each clause is a group of 3 nodes
  - Connect all nodes except:
    - Contradictory nodes
    - Nodes in the same group

Don't forget iff

=> If $\phi \in 3SAT$
  - Then each clause has a TRUE literal
    - Those are nodes in the clique!
    - eg $x_1 = 0, x_2 = 1$

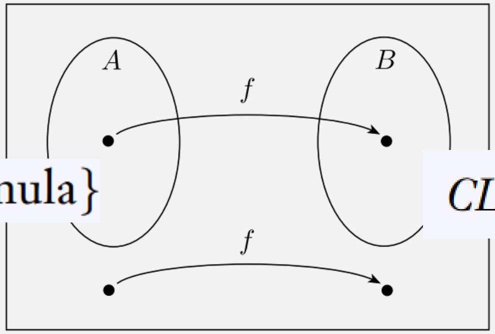
<= If f(ϕ) in *CLIQUE*
  - Each node of clique must come from different group
  - So original formula is satisfiable, by because each group can be made TRUE



Must show fn runs in **polynomial time:**
- # literals = # nodes $O(k)$
- # edges poly in # nodes $O(k^2)$

Theorem: *3SAT* is polynomial time reducible to *CLIQUE*.

$3SAT = \{\langle\phi\rangle \mid \phi \text{ is a satisfiable 3cnf-formula}\}$



$CLIQUE = \{\langle G, k\rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$

- So since *CLIQUE* is in **NP**, then *3SAT* is also in **NP**

NP-Completeness

Must prove for all langs, not just one single language

DEFINITION 7.34

A language B is ***NP-complete*** if it satisfies two conditions:

1. B is in NP, and
2. every A in NP is polynomial time reducible to B.

easy

hard???? ...

- How does this help the **P** = **NP** problem?

... figuring out the first NP-Complete problem is hard!

THEOREM 7.35

If B is NP-complete and $B \in \mathrm{P}$, then $\mathrm{P} = \mathrm{NP}$

(Just like figuring out the first undecidable problem was hard!)

But then we use that problem to prove other problems NP-Complete!

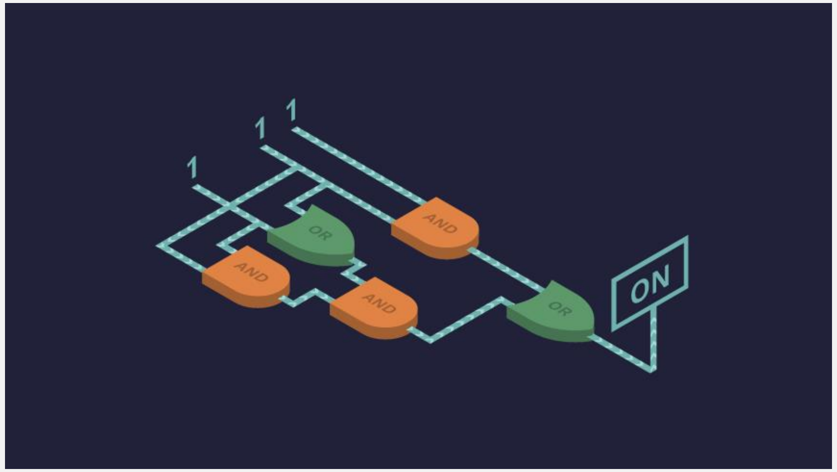
Next time: The Cook-Levin Theorem

The first NP-Complete problem

THEOREM 7.37

SAT is NP-complete

But it makes sense that every problem can be reduced to it ...



Check-in Quiz 4/28

On gradescope