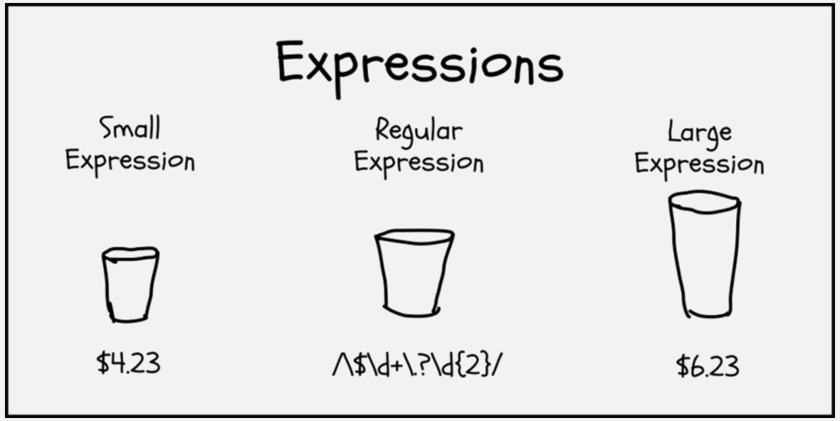
UMB CS 420

Regular Expressions

Wednesday, February 9, 2022

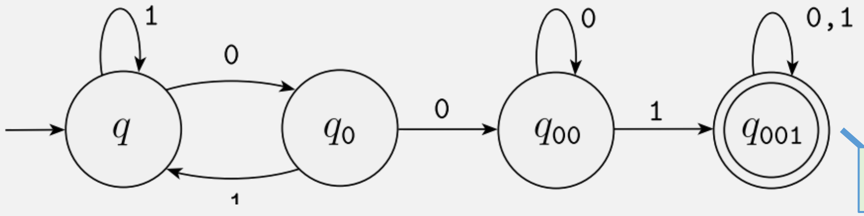


Announcements

- HW 2 due Sunday 2/13 11:59pm EST

So Far: Regular Language Representations

1. State diagram (NFA/DFA)



(it doesn't fit!)

A practical application: text search

2. Formal description

1. $Q = \{q_1, q_2, q_3\}$,
2. $\Sigma = \{0,1\}$,
3. δ is described as

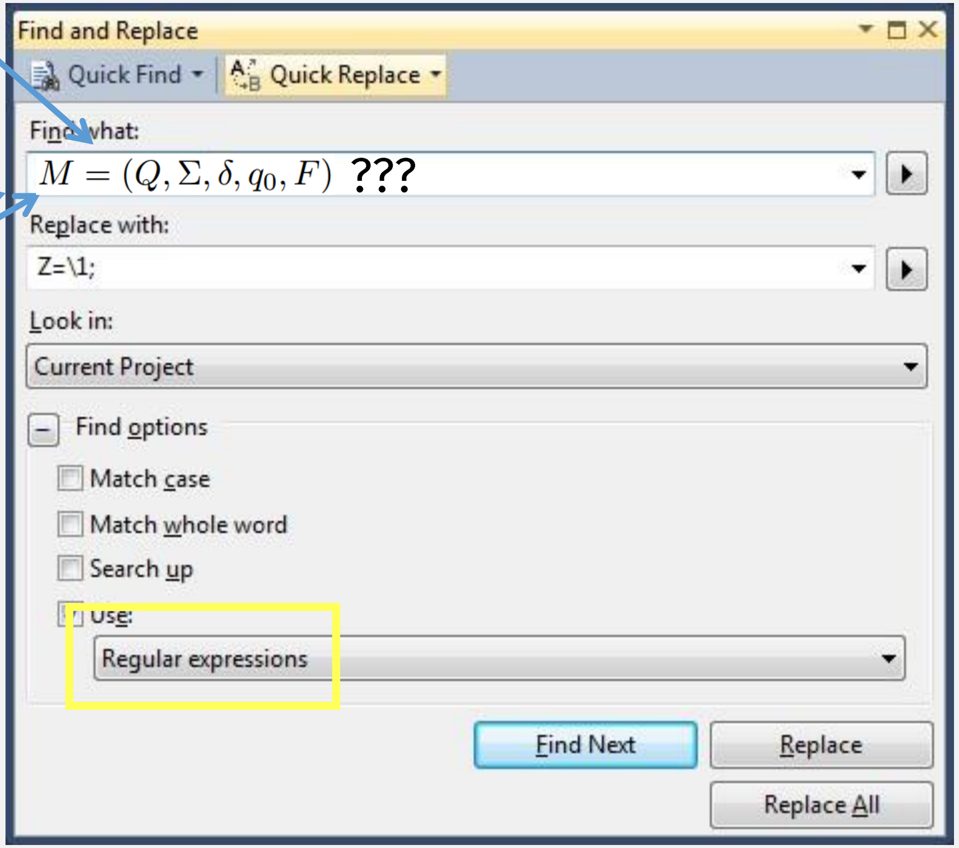
	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2,

4. q_1 is the start state, and
5. $F = \{q_2\}$.

These define a computer (program) that finds strings containing `001`

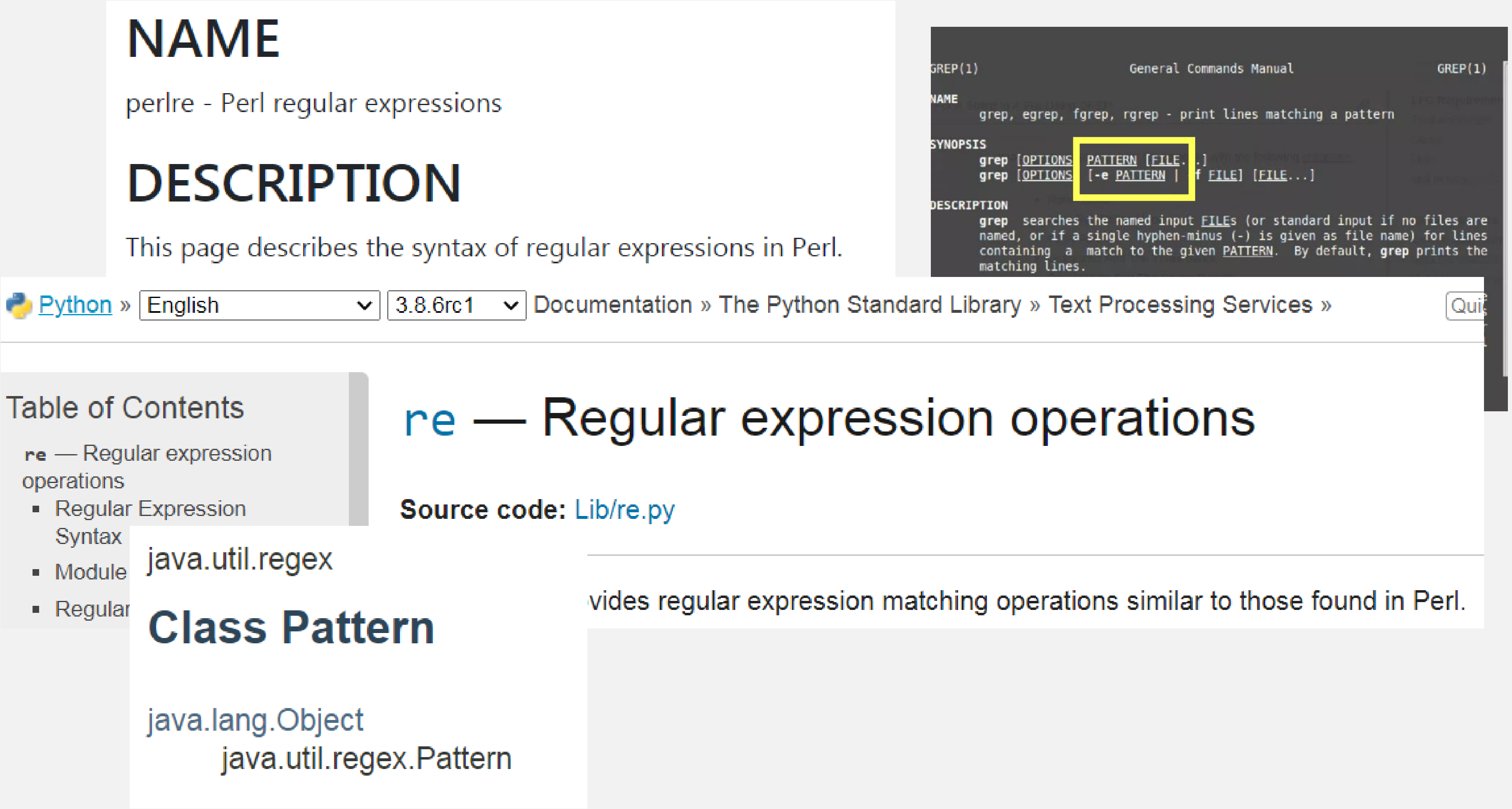
3. $\Sigma^*001\Sigma^*$

Need a more concise (textual) notation



Regular Expressions Are Widely Used

- Unix
- Perl
- Python
- Java



Last Time: Why Do We Care These Ops Are Closed?

- Union
- Concat
- Kleene star

- The are sufficient to represent all regular languages!

- I.e., they are used to define **regular expressions**

Regular Expressions: Formal Definition

R is a ***regular expression*** if R is

1. a for some a in the alphabet Σ,

2. ε,

3. $\emptyset$,

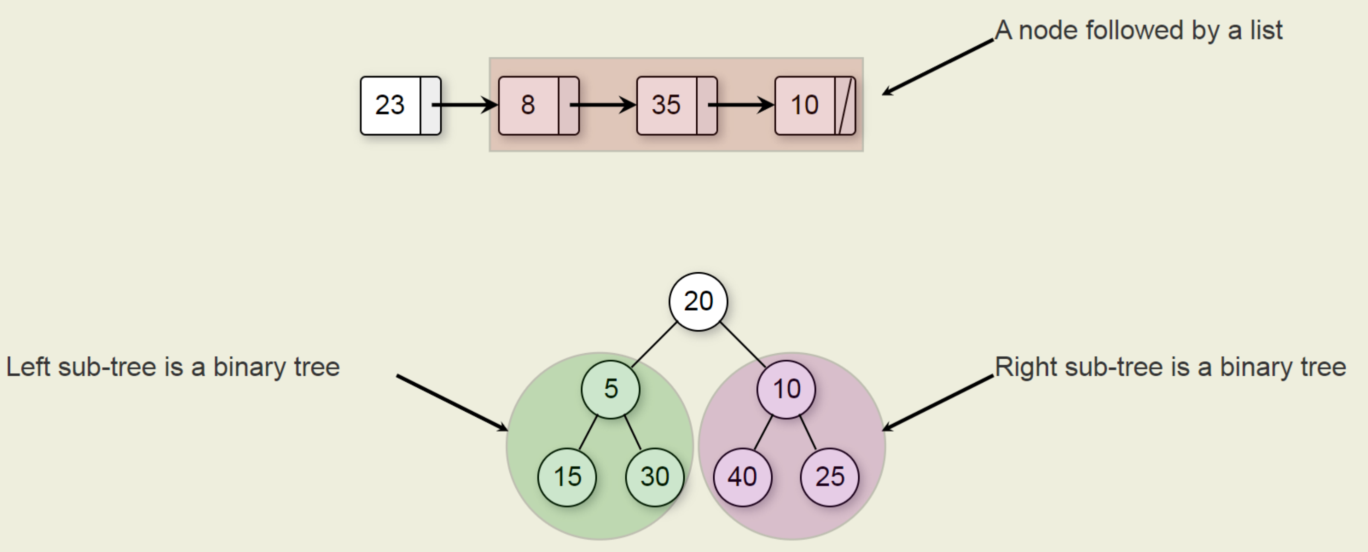
4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,

5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or

6. (R_1^*), where R_1 is a regular expression.

This is a <u>recursive definition</u>

Recursive Definitions



Recursive definitions have:
- base case and
- recursive case
  (with a "smaller" object)

```
/* Linked list Node*/
class Node {
    int data;
    Node next;
}
```

This is a recursive definition:
Node used before it's defined
(but must be "smaller")

Regular Expressions: Formal Definition

R is a ***regular expression*** if R is

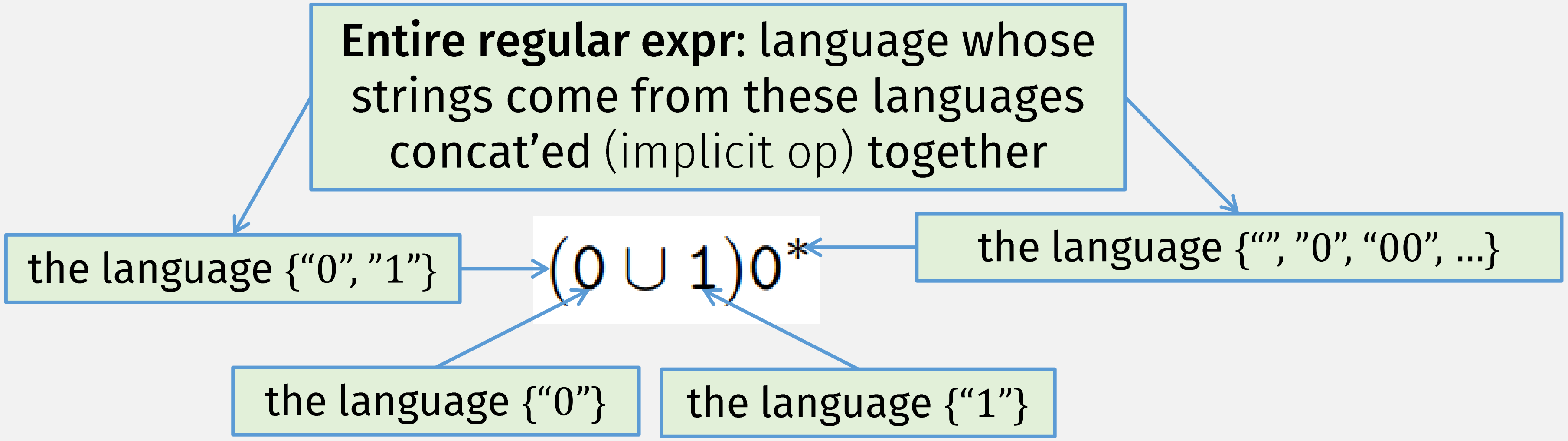
3 Base Cases

1. a for some a in the alphabet Σ, (A lang containing a) **length-1 string**
2. ε, (A lang containing) **the empty string**
3. $\emptyset$, **The empty set** (i.e., a lang containing no strings)

3 Recursive Cases

4. (union) $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
5. (concat) $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
6. (star) (R_1^*), where R_1 is a regular expression.

Regular Expression: Concrete Example



- Operator Precedence:
  - Parentheses
  - Kleene Star
  - Concat (sometimes ∘, sometimes implicit)
  - Union

R is a ***regular expression*** if R is

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Regular Expressions = Regular Langs?

R is a ***regular expression*** if R is

3 Base Cases

1. a for some a in the alphabet Σ,
2. ε,
3. $\emptyset$,

3 Recursive Cases

4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
6. (R_1^*), where R_1 is a regular expression.

Base cases + union, concat, and Kleene star can express <u>any regular language</u>!

(But we have to prove it)

Thm: A Lang is Regular **iff** Some Reg Expr Describes It

⇒ If a language is regular, it is described by a reg expression

⇐ If a language is described by a reg expression, it is regular

- Easier
- **For a given regular expression, convert to equiv NFA!**
- (Hint: we mostly did this already when discussing closed ops)

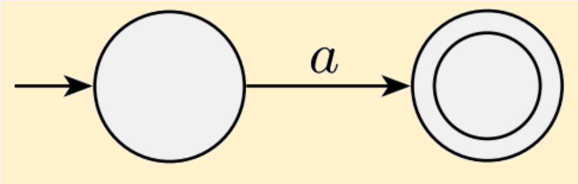
How to show that a language is regular?

Construct a DFA *or* NFA!

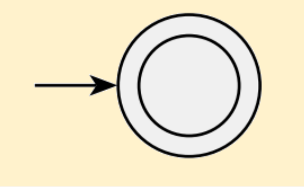
RegExpr→NFA

R is a ***regular expression*** if R is

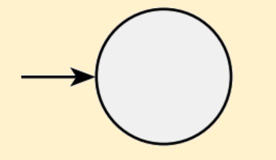
1. a for some a in the alphabet Σ,



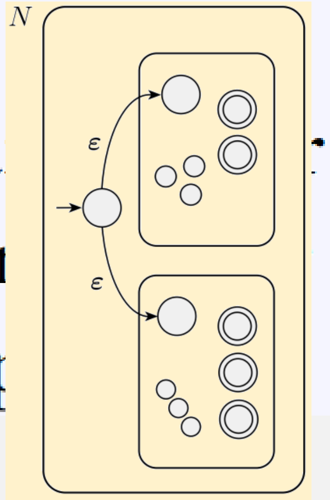
2. ε,

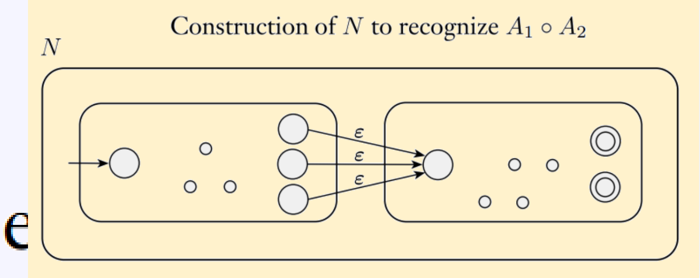


3. $\emptyset$,



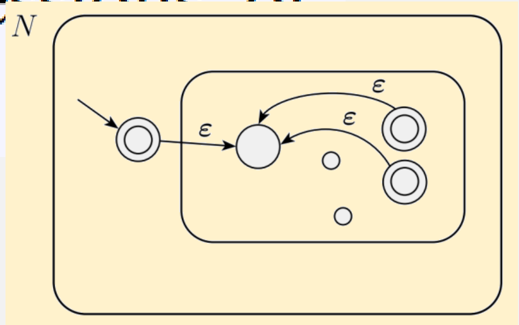
4. $(R_1 \cup R_2)$, where R_1 and R_2 a





5. $(R_1 \circ R_2)$, where R_1 and R_2 ar expressions, or

6. (R_1^*), where R_1 is a regular exp



Thm: A Lang is Regular **iff** Some Reg Expr Describes It

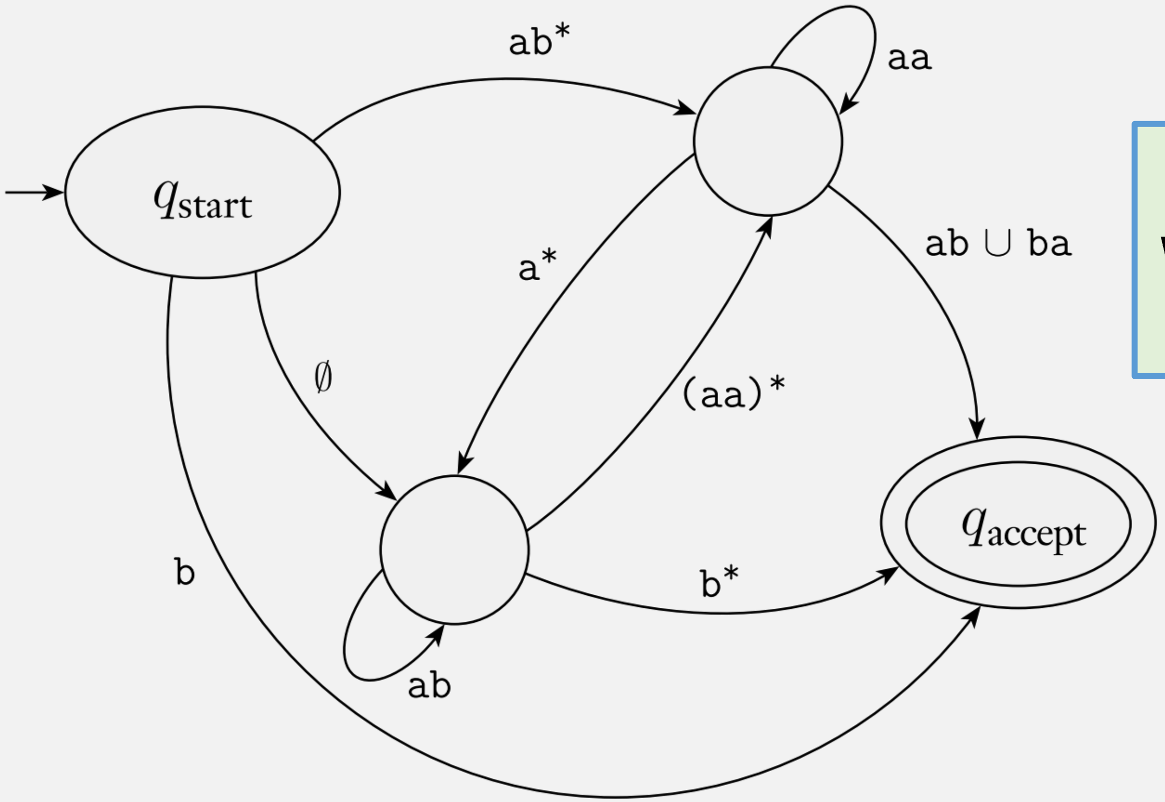
⇒ If a language is regular, it is described by a reg expression

- Harder
- Need to convert an DFA or NFA to an equivalent Regular Expression
- To do so, we need another kind of finite automata: a **GNFA**

⇐ If a language is described by a reg expression, it is regular

- Easier
- ☑ Convert the regular expression to an equivalent NFA!

Generalized NFAs (GNFAs)



A regular NFA is a GNFA with only single character regular expr transitions

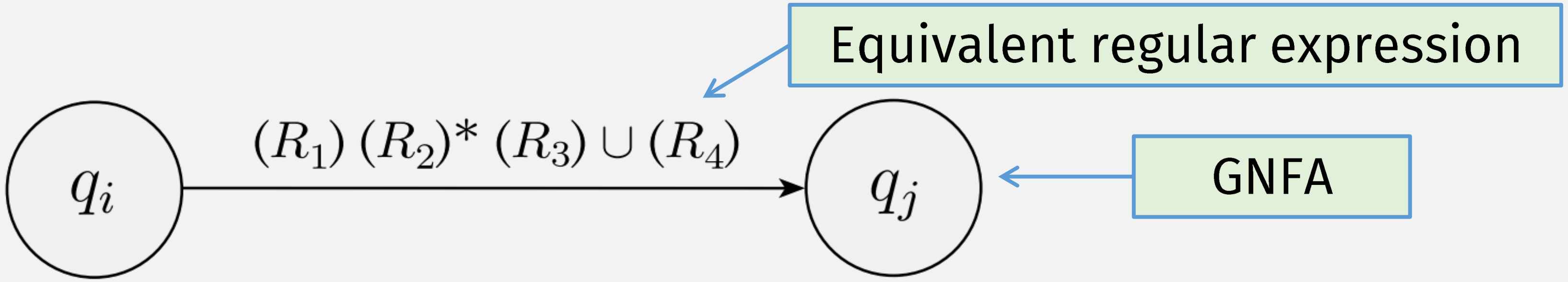
Goal: **convert GNFAs to Regular Exprs**

- GNFA = NFA with regular expression transitions

GNFA→RegExpr function

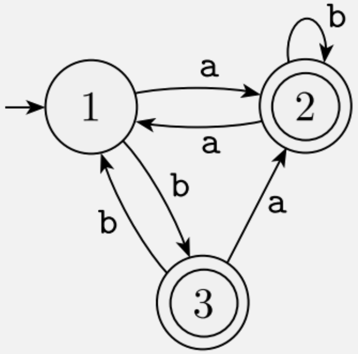
On GNFA input G:

- If G has 2 states, return the regular expression transition, e.g.:



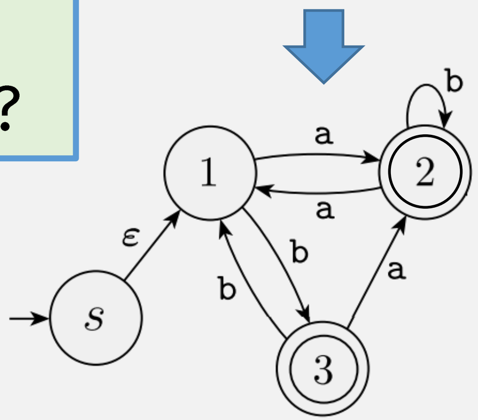
Could there be less than 2 states?

GNFA→RegExpr Preprocessing

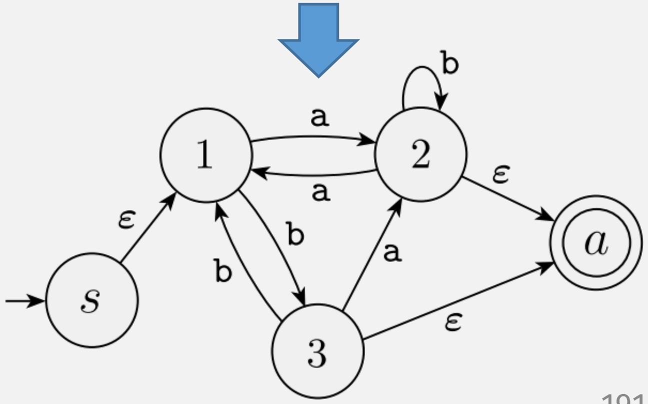


- First modifies input machine to have:

Does this change the language of the machine?



  - New start state:
    - No incoming transitions
    - ε transition to old start state



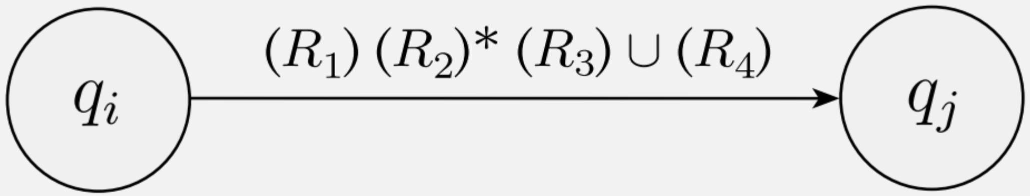
  - New, single accept state:
    - With ε transitions from old accept states

GNFA→RegExpr function (recursive)

On GNFA input G:

Base Case

- If G has 2 states, return the regular expression transition, e.g.:

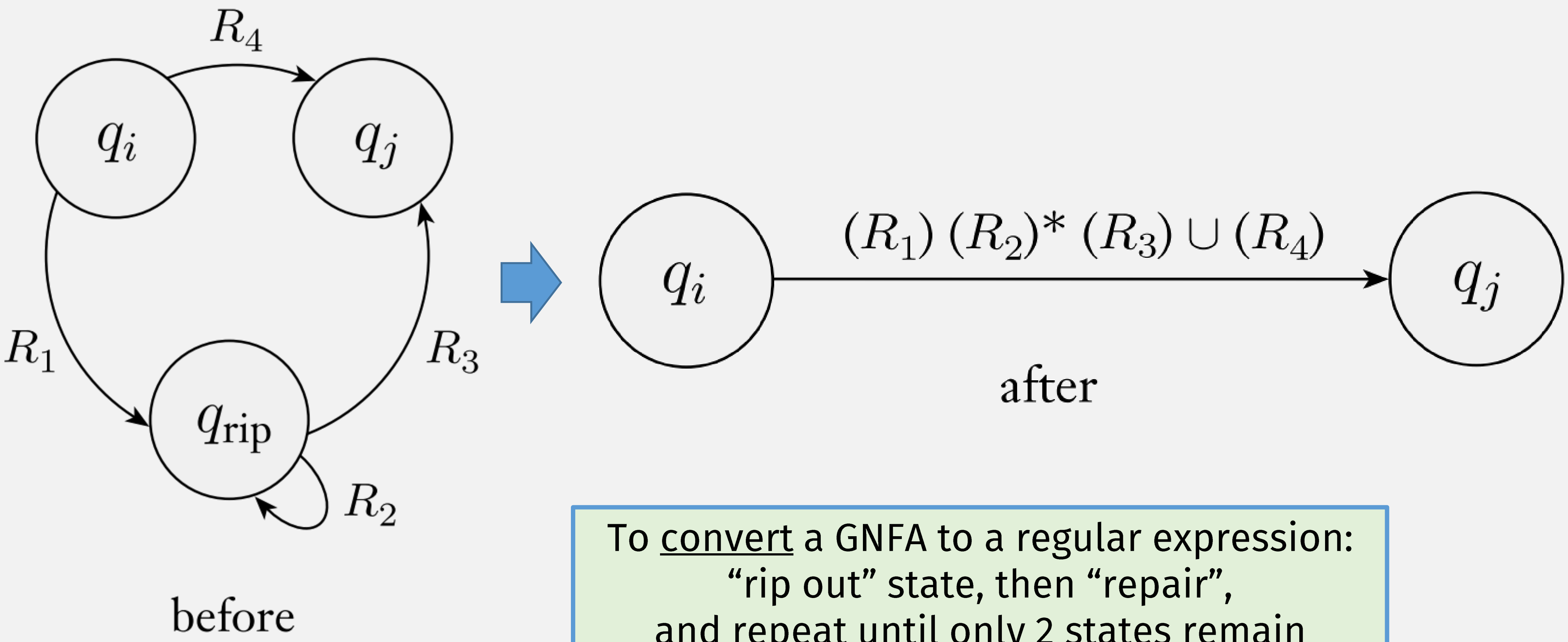


Recursive Case

- Else:
  - "Rip out" one state
  - "Repair" the machine to get an equivalent GNFA G'
  - Recursively call GNFA→RegExpr(G')

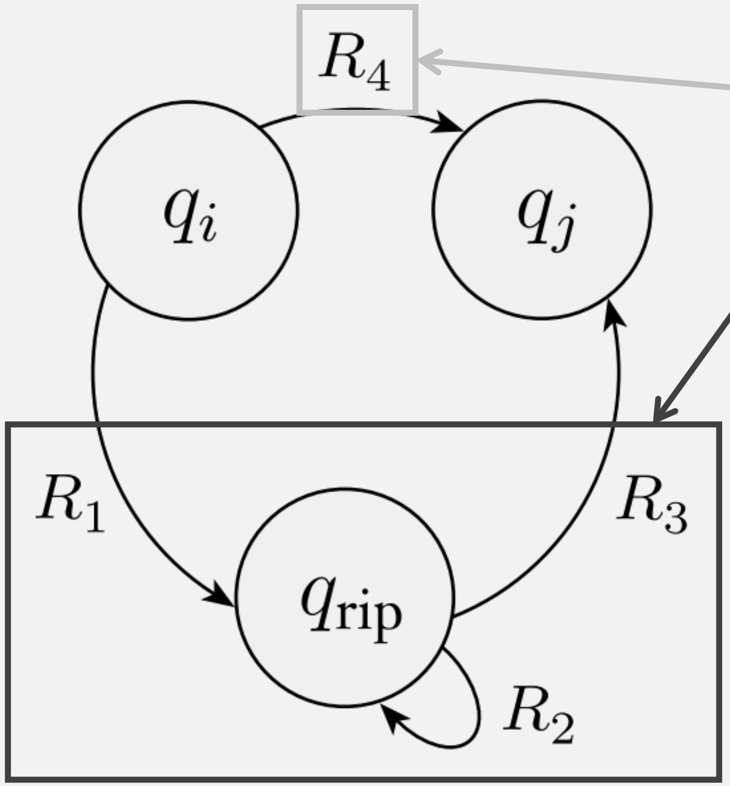
Recursive definitions have:
- base case and
- recursive case
  (with a "smaller" object)

GNFA→RegExpr: "Rip/Repair" step



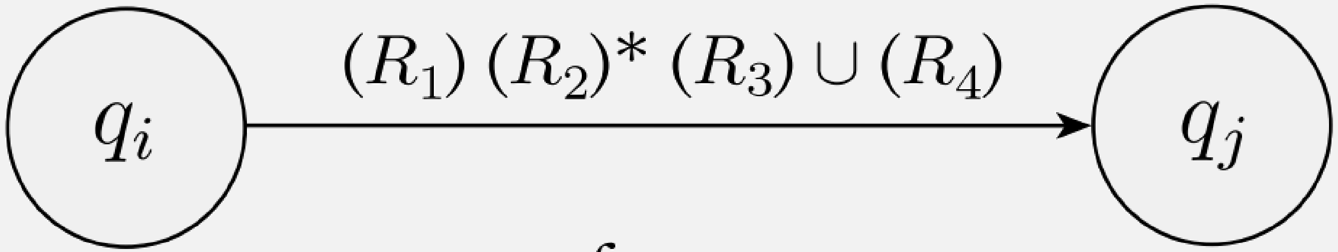
To <u>convert</u> a GNFA to a regular expression:
"rip out" state, then "repair",
and repeat until only 2 states remain

GNFA→RegExpr: "Rip/Repair" step



before

<u>Before</u>: two paths from q_i to q_j:
1. Not through q_{rip}
2. Through q_{rip}

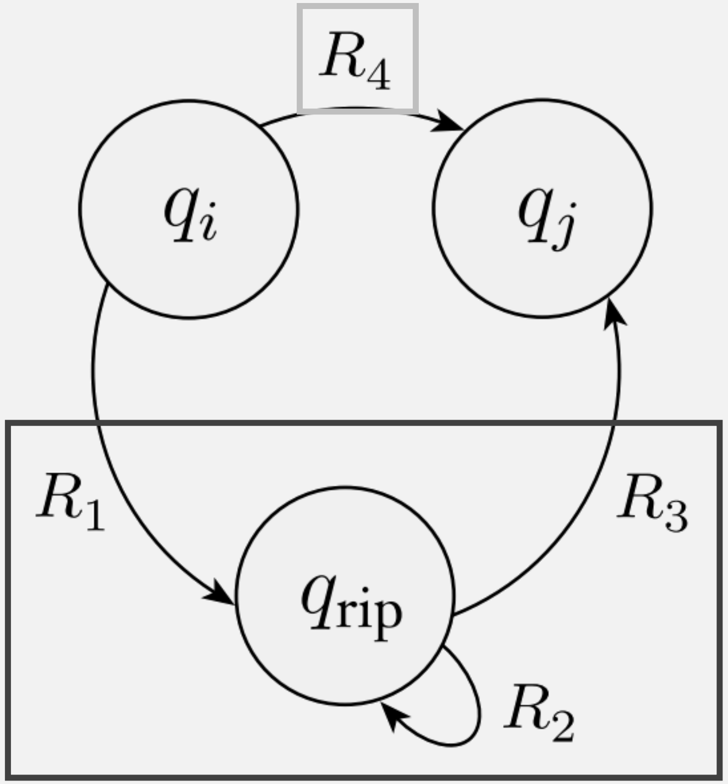


after

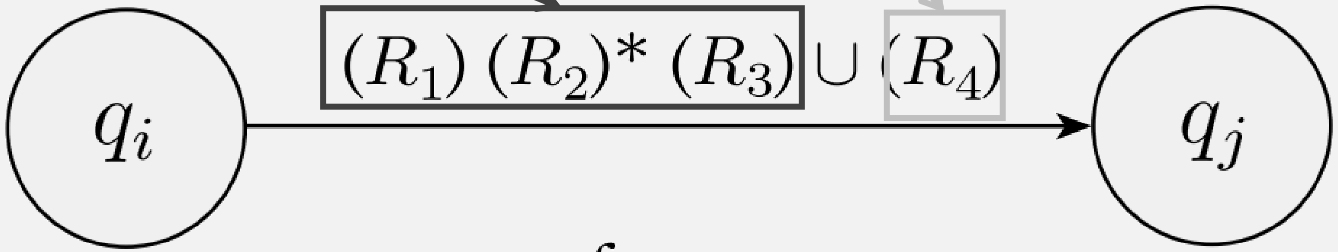
GNFA→RegExpr: "Rip/Repair" step

After: still two "paths" from q_i to q_j

1. Not through q_{rip}
2. Through q_{rip}

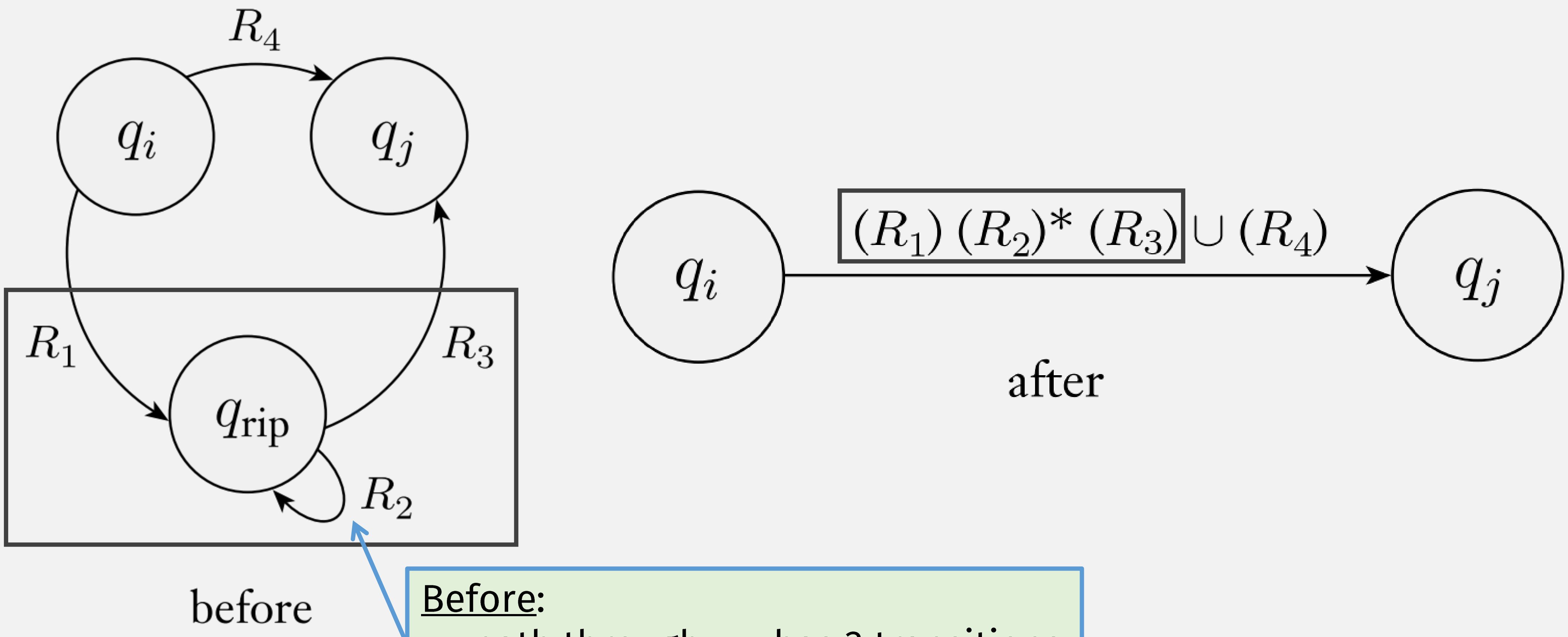


before



after

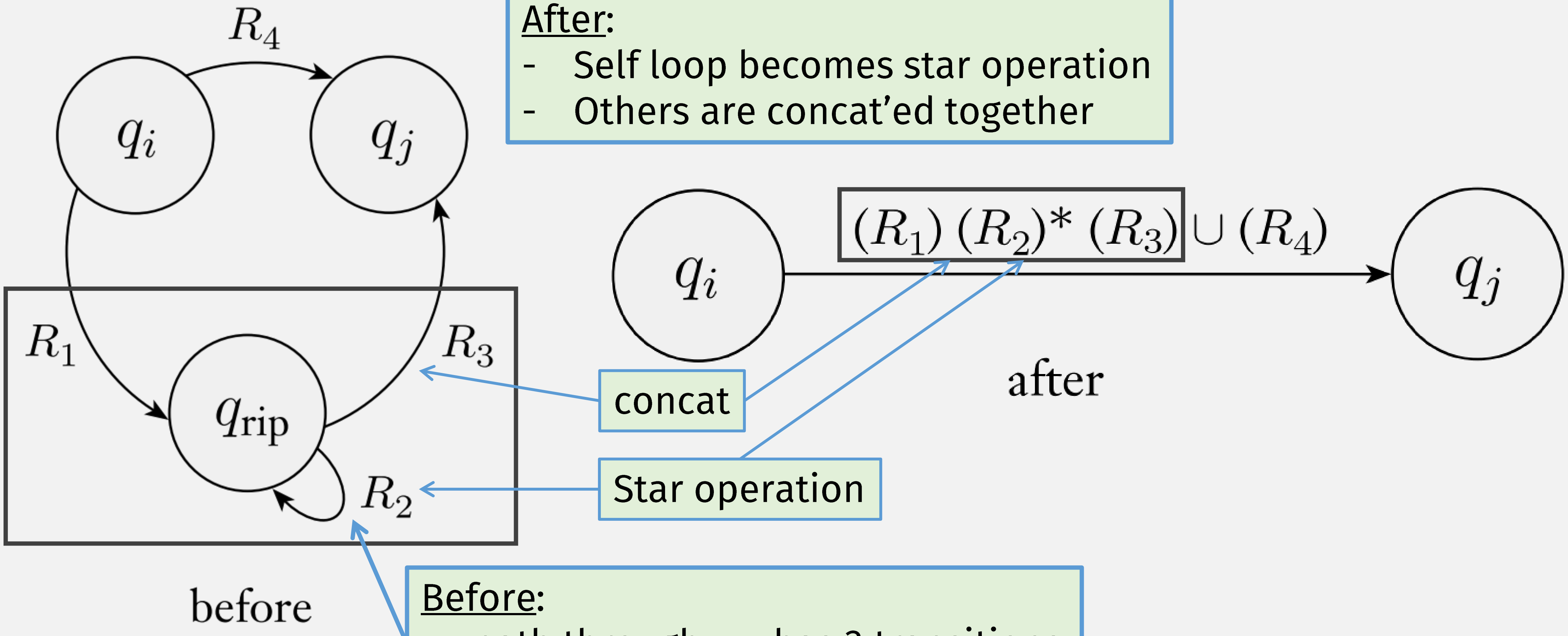
GNFA→RegExpr: "Rip/Repair" step



<u>Before</u>:
- path through q_{rip} has 3 transitions
- One is self loop

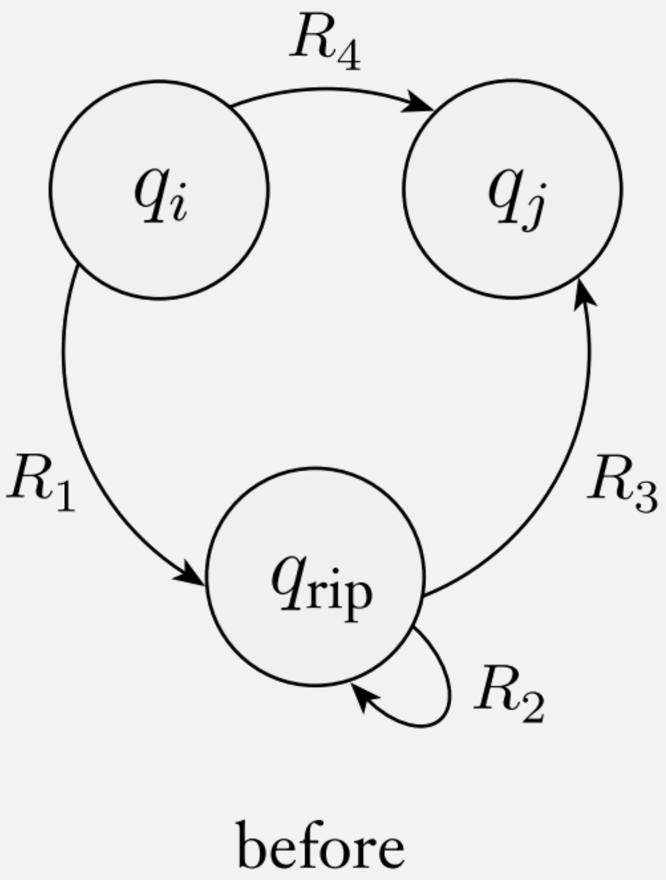
GNFA→RegExpr: "Rip/Repair" step

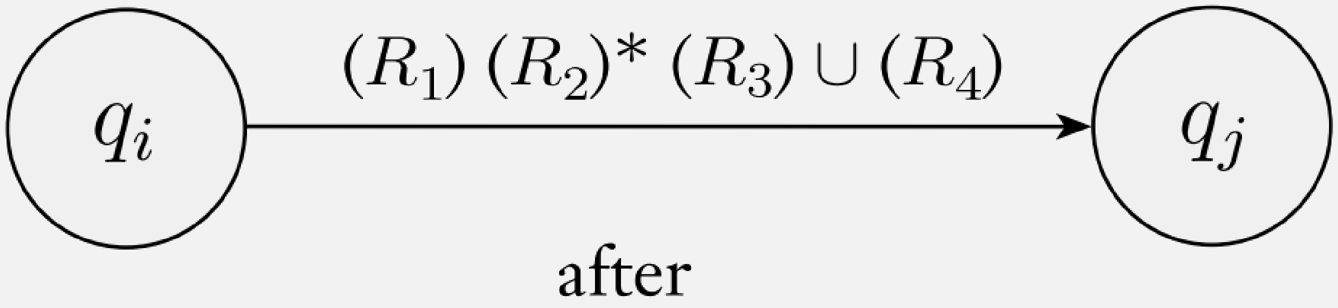
After:
- Self loop becomes star operation
- Others are concat'ed together



Before:
- path through q_{rip} has 3 transitions
- One is self loop

GNFA→RegExpr: Rip/Repair "Correctness"





Must show these are <u>equivalent</u>

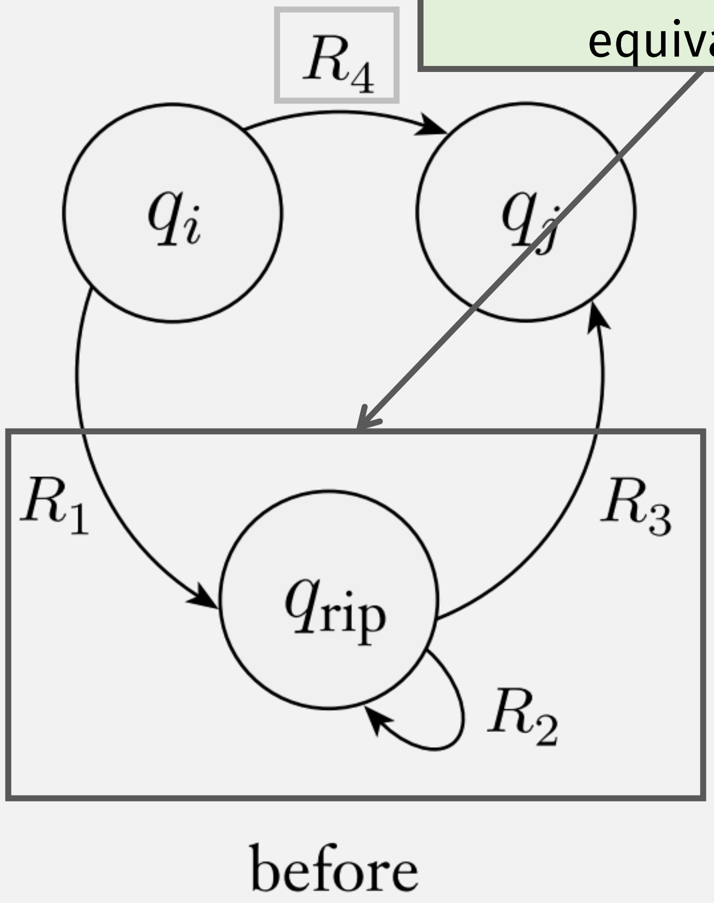
GNFA→RegExpr "Correctness"

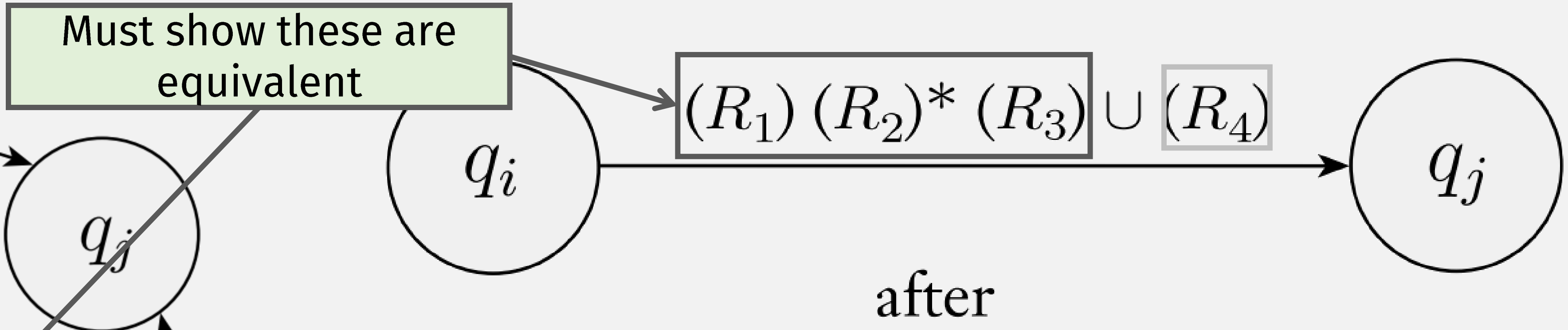
- Where "Correct" / "Equivalent" means:

$$\text{LangOf}\,(\,G\,) = \text{LangOf}\,(\,\textbf{GNFA→RegExpr}(\,G\,)\,)$$

- i.e., **GNFA→RegExpr** must not change the language!
  - Key step: the rip/repair step

GNFA→RegExpr: Rip/Repair "Correctness"





Must <u>prove</u>:

- Every string accepted <u>before</u>, is accepted <u>after</u>
- <u>2 cases:</u>
    1. Accepted string does not go through q_{rip}
        - Acceptance unchanged (both use R_4 transition part)
    2. String goes through q_{rip}
        - Acceptance unchanged?
        - Yes, via our previous reasoning

Thm: A Lang is Regular **iff** Some Reg Expr Describes It

⇒ If a language is regular, it is described by a regular expr

- Need to convert DFA or NFA to Regular Expression
- ☑ Use **GNFA→RegExpr** to convert GNFA to equiv regular expression!

⇐ If a language is described by a regular expr, it is regular

- ☑ Convert regular expression to equiv NFA!

∎

Now we may use regular expressions to represent regular langs.

So we also have another way to prove things about regular languages!

So a regular language has these equivalent representations:
- DFA
- NFA
- Regular Expression

How to Prove A Language Is Regular?

- Construct DFA
- Construct NFA
- Create Regular Expression

Slightly different because of recursive definition

R is a ***regular expression*** if R is

1. a for some a in the alphabet Σ,
2. ε,
3. $\emptyset$,
4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
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6. (R_1^*), where R_1 is a regular expression.

Kinds of Mathematical Proof

- Proof by construction
- Proof by induction
  - Use this when working with recursive definitions

Proof by Induction

To prove that a ***Statement*** is true for a **recursively defined thing** x:

1. Prove *Statement* for the base case of x (usually easy)
2. Prove *Statement* for the inductive (recursive) case of x:
   - Assume the induction hypothesis (IH):
     - I.e., *Statement* is true for some "smaller" x_{smaller}
     - E.g., if x is string, then "smaller" = length of string
   - Use IH (and other facts) to prove *Statement* for "larger" x
     - Usually involves a case analysis on how to go from x_{smaller} to x

- Why can we assume IH is true???
  - Because we can always start at base case,
  - Then use it to prove for slightly larger case,
  - Then use that to prove for slightly larger case …

Natural Numbers Are Recursively Defined

A Natural Number is:

- zero
- Or $n + 1$, where n is a Natural Number

This definition is valid because recursive reference is “smaller”

So proving things about Natural Numbers requires induction!

Proof By Induction: Example (Sipser Ch 0)

Prove true: $P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right)$

- P_t = loan balance after t months
- t = # months
- P = principal = original amount of loan
- M = interest (multiplier)
- Y = monthly payment

Proof By Induction: Example (Sipser Ch 0)

Prove true: $$P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right)$$

Proof: by **induction** on natural number t

An inductive proof exactly follows the recursive definition (here, natural numbers) that the induction is "on"

Base Case, $t = 0$:

- Goal: **Show** $P_0 = P$
- Proof of Goal:

$$P_0 = PM^0 - Y\left(\frac{M^0 - 1}{M - 1}\right) = P$$

Plug in $t = 0$

Simplify, to get to goal statement

A Natural Number is:

- zero
- Or $n + 1$, where n is a natural number

Proof By Induction: Example (Sipser Ch 0)

An inductive proof exactly follows the recursive definition (here, natural numbers) **that the induction is "on"**

A Natural Number is:

- zero
- Or $n + 1$, where n is a natural number

Prove true: $$P_t = PM^t - Y\left(\frac{M^t - 1}{M - 1}\right)$$

Inductive Case: $t > 0$

- Inductive Hypothesis (IH), **assume statement true for some** $t = k$

$$P_k = PM^k - Y\left(\frac{M^k - 1}{M - 1}\right)$$

"Connect together" known definitions and statements

- Goal **statement to prove, for** $t = k+1$:

$$P_{k+1} = PM^{k+1} - Y\left(\frac{M^{k+1} - 1}{M - 1}\right)$$

Plug in IH

Simplify, to derive goal statement

- Proof of Goal:

$$P_{k+1} = P_k M - Y = \left[PM^k - Y\left(\frac{M^k - 1}{M - 1}\right)\right] M - Y = PM^{k+1} - Y\left(\frac{M^{k+1} - 1}{M - 1}\right)$$

Definition of P_{k+1}

Homomorphisms

A ***homomorphism*** is a function $f : \Sigma \longrightarrow \Gamma$ from one alphabet to another.

- Assume *f* can be used on characters, strings, and languages
- E.g., like a secret decoder!
  - *f* ("x") -> "c"
  - *f* ("y") -> "a"
  - *f* ("z") -> "t"
  - *f* ("xyz") -> "cat"

Homomorphisms Closed Under Regular Languages

Thm: If a language A is regular, and f is a homomorphism, ...

... then $f(A)$ is a regular language

A ***homomorphism*** is a function $f: \Sigma \longrightarrow \Gamma$ from one alphabet to another.

How to Prove A Language Is Regular?

- Construct DFA
- Construct NFA
- Create Regular Expression

Slightly different because of recursive definition

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Homomorphisms Closed Under Regular Languages

Thm: If a language A is regular, and f is a homomorphism, …

- If A is regular then it has a regular expression R

… then $f(A)$ is a regular language

- To show that $f(A)$ is a regular language, we create a regular expression representing it (using R)

A ***homomorphism*** is a function $f: \Sigma \longrightarrow \Gamma$ from one alphabet to another.

Homomorphisms Closed Under Regular Languages

Thm: If language A is regular, and f is a homomorphism, then $f(A)$ is regular

Proof: By induction on R, the regular expression for A

An inductive proof exactly follows the recursive definition that the induction is "on"

R is a ***regular expression*** if R is

3 Base Cases

1. a for some a in the alphabet Σ,
2. ε,
3. $\emptyset$,

If $R = a$, then $f(A)$ has a regular expression $f(a)$ and is thus regular

3 Recursive Cases

4. $(R_1 \cup R_2)$, where R_1 and R_2 are regular expressions,
5. $(R_1 \circ R_2)$, where R_1 and R_2 are regular expressions, or
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3 Recursive Cases

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Inductive case # 1: $R = R_1 \cup R_2$

Homomorphisms Closed Under Regular Languages

Thm: If language A is regular, and f is a homomorphism, then $f(A)$ is regular

Proof: By induction on R, the regular expression for A

Inductive Case #1: $R = R_1 \cup R_2$ where R_1, R_2 describe "smaller" reg langs A_1, A_2

IH (assume the theorem is true for "smaller" languages)

- If language A_1 is regular, then $f(A_1)$ is regular
- If language A_2 is regular, then $f(A_2)$ is regular

Goal: If language $A_1 \cup A_2$ is regular, then $f(A_1 \cup A_2)$ is regular

Proof of Goal **(piece together known definitions and statements!)**

- $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$ (because f and $\cup$ don't affect each other)
- $f(A_1)$ is regular (because of IH)
- $f(A_2)$ is regular (because of IH)
- $f(A_1) \cup f(A_2)$ is regular (because union is closed for regular languages)

In-Class quiz 2/9

See gradescope