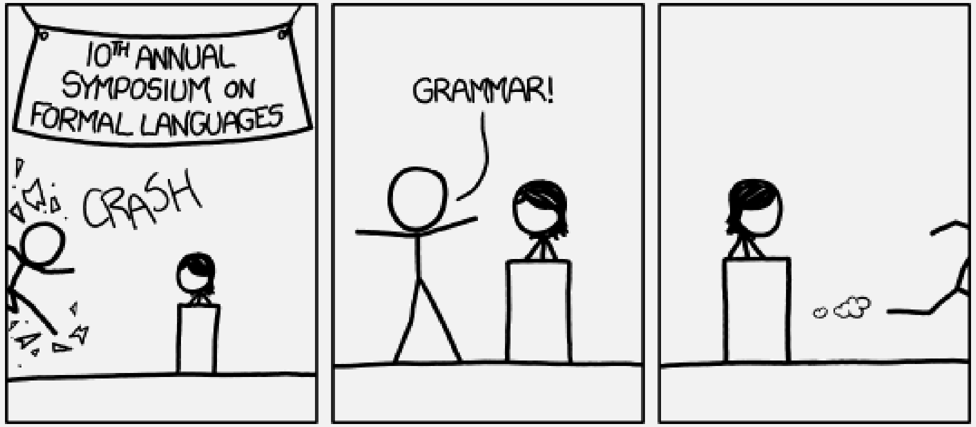
UMB CS 420

Pushdown Automata (PDAs)

Wednesday, February 23, 2022



Announcements

- HW 3 in

- HW 4 out
  - Due Sun 2/27 11:59pm EST

Last Time: Generating Strings with a CFG

A **CFG** represents a **context free language!**

$$G_1 =$$

$$A \to 0A1$$
$$A \to B$$
$$B \to \#$$

Strings in CFG's language
= all possible generated strings

$L(G_1)$ is $\{0^n\#1^n \mid n \geq 0\}$

Stop when string is all terminals

A CFG **generates** a string, by repeatedly applying substitution rules:

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$$

Start variable

Last Time:

Regular Languages	**Context-Free** Languages (CFLs)
Regular Expression	Context-Free Grammar (CFG)
A Reg Expr <u>describes</u> a Regular lang	A CFG <u>describes</u> a CFL

Today:

Regular Languages	**Context-Free** Languages (CFLs)
Regular Expression	Context-Free Grammar (CFG)
A Reg Expr describes a Regular lang	A CFG describes a CFL
TODAY:	
Finite Automaton (FSM)	Push-down automaton (PDA)
An FSM recognizes a Regular lang	A PDA recognizes a CFL

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Regular Languages	**Context-Free** Languages (CFLs)
Regular Expression	Context-Free Grammar (CFG)
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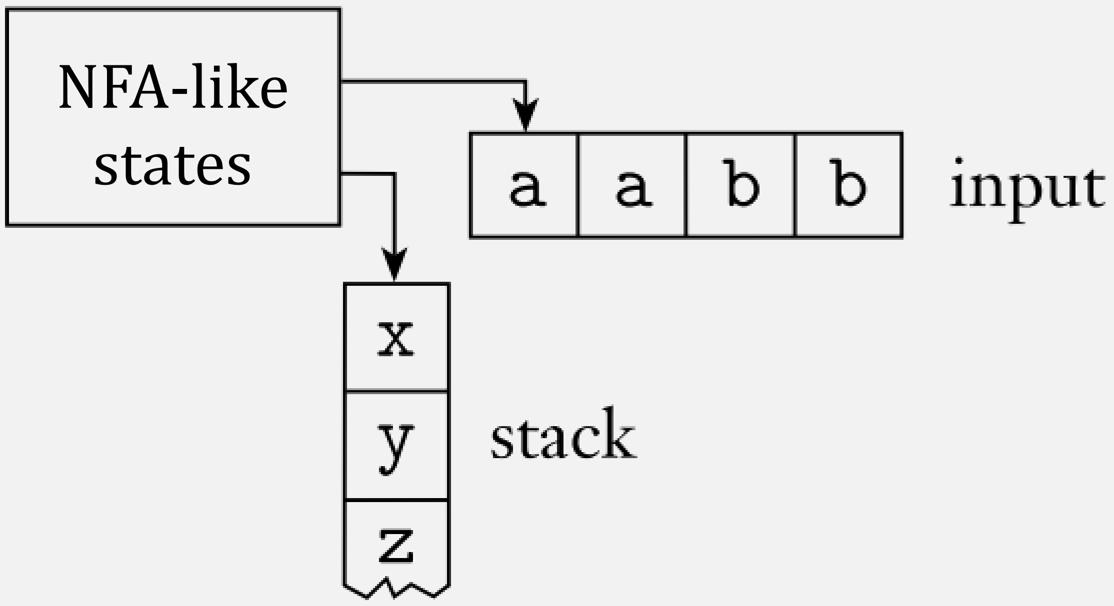
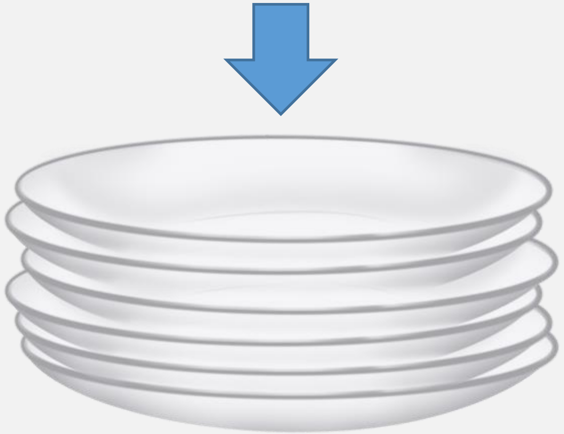
TODAY:	
Finite Automaton (FSM)	Push-down automaton (PDA)
An FSM recognizes a Regular lang	A PDA recognizes a CFL
KEY DIFFERENCE:	
A Regular lang is defined with a FSM	A CFL is defined with a CFG
Must prove: Reg Expr ⇔ Reg lang	*Must prove*: PDA ⇔ CFL

Pushdown Automata (PDA)

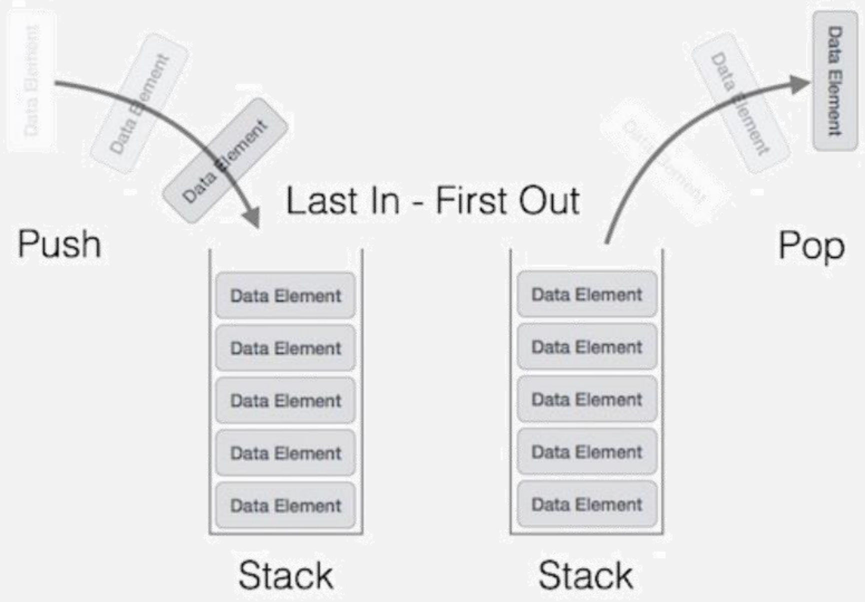
PDA = NFA + a stack



What is a Stack?

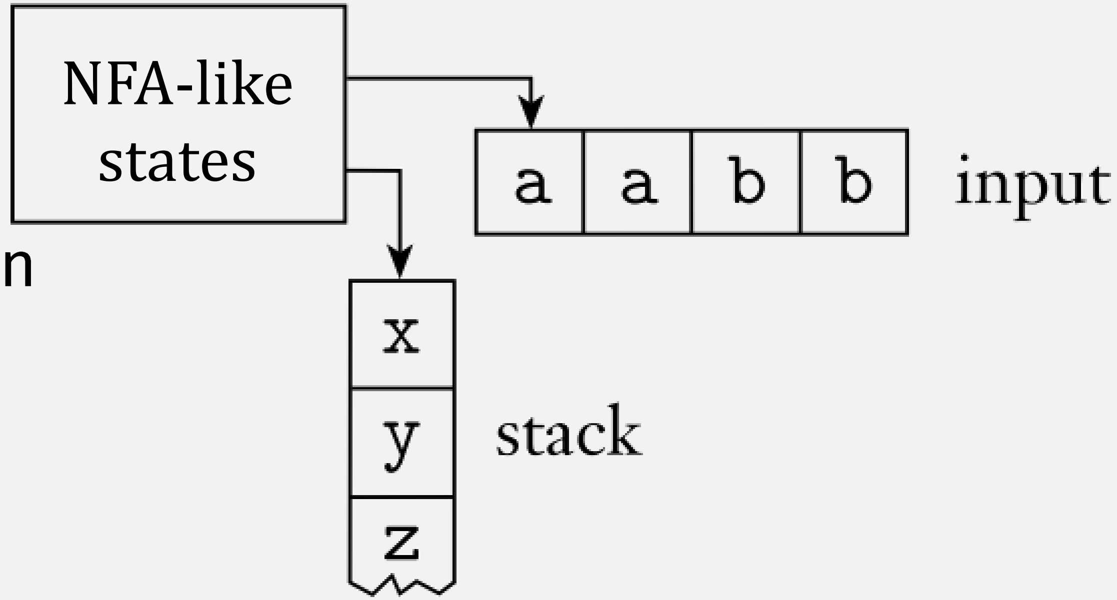


- A <u>restricted</u> kind of (infinite) memory
- **Access to top element only**
- 2 Operations only: **push, pop**



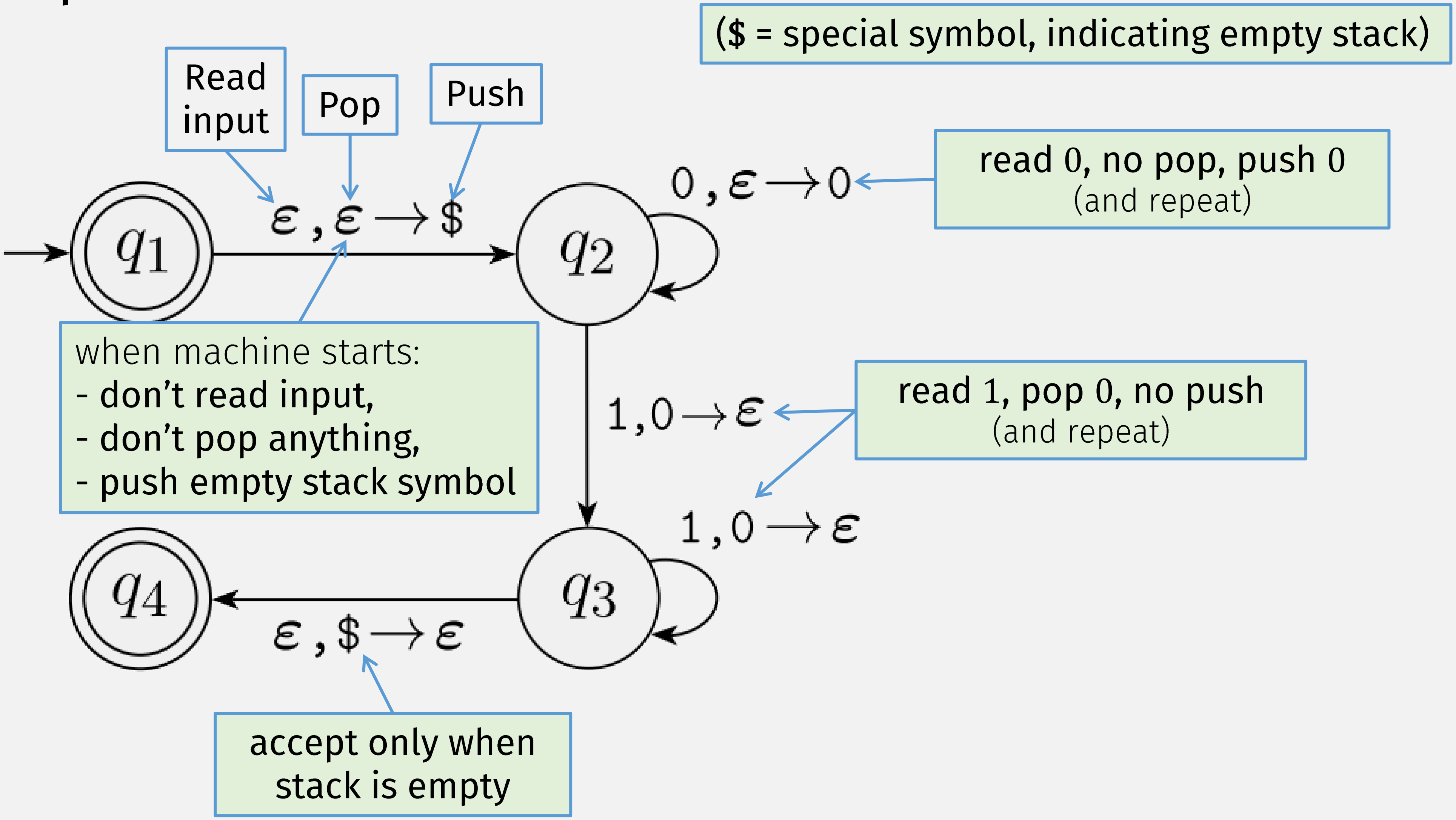
Pushdown Automata (PDA)

- PDA = NFA + a stack
  - Infinite memory
  - Can only read/write top location
    - Push/pop



An Example PDA

$\{0^n 1^n \mid n \geq 0\}$



Formal Definition of PDA

A ***pushdown automaton*** is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$, where Q, Σ, Γ, and F are all finite sets, and

1. Q is the set of states,
2. Σ is the input alphabet,
3. Γ is the stack alphabet,
4. $\delta \colon Q \times \Sigma_\varepsilon \times \Gamma_\varepsilon \longrightarrow \mathcal{P}(Q \times \Gamma_\varepsilon)$ is the transition function,
5. $q_0 \in Q$ is the start state, and
6. $F \subseteq Q$ is the set of accept states.

Stack alphabet can have special stack symbols, e.g., **$**

Input

Pop

Push

Non-deterministic: produces a **set** of (STATE, STACK CHAR) pairs

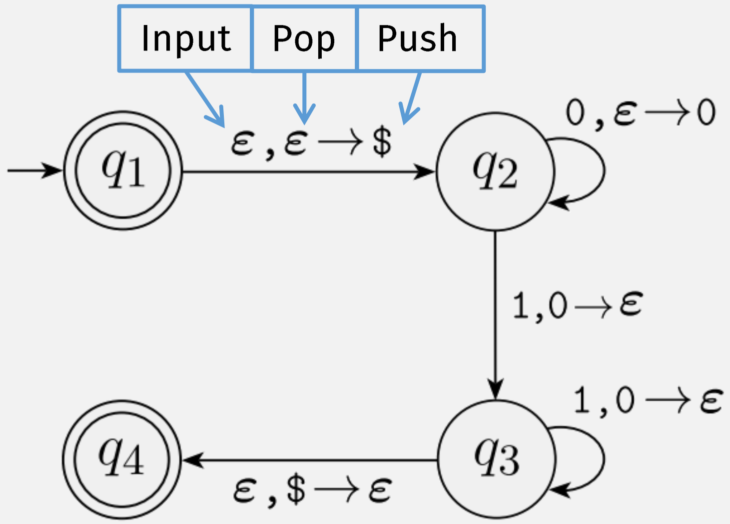
PDA Formal Definition Example

$$Q = \{q_1, q_2, q_3, q_4\},$$

$$\Sigma = \{0,1\},$$

$$\Gamma = \{0, \$\},$$

$$F = \{q_1, q_4\},$$



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Input Pop Push

$Q = \{q_1, q_2, q_3, q_4\}$,

$\Sigma = \{0,1\}$,

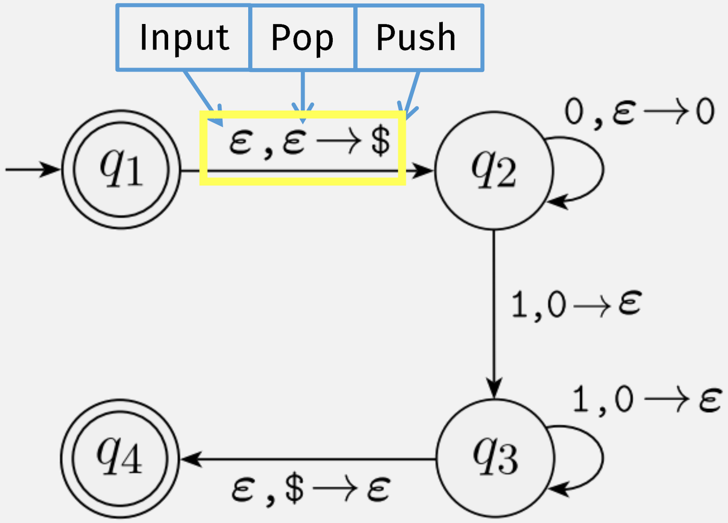
$\Gamma = \{0, \$\}$,

$F = \{q_1, q_4\}$, and

δ is given by the following table, wherein blank entries signify $\emptyset$.

Input:	0			1			ε		
Stack:	0	\$	ε	0	\$	ε	0	\$	ε
q_1									$\{(q_2, \$)\}$ 5
q_2			$\{(q_2, 0)\}$ 1	$\{(q_3, \varepsilon)\}$ 2				4	
q_3				$\{(q_3, \varepsilon)\}$ 3				$\{(q_4, \varepsilon)\}$	
q_4									

Input — Pop — Push



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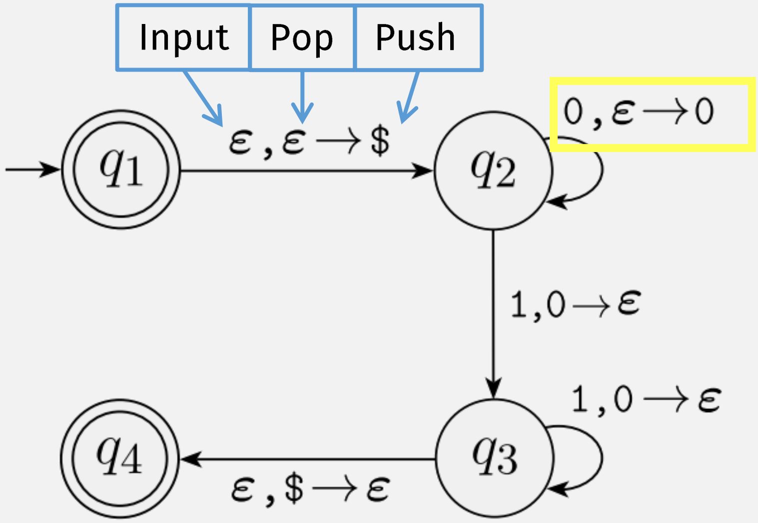
Input:	0			1			ε			Input
Stack:	0	\$	ε	0	\$	ε	0	\$	ε	Pop
q_1									$\{(q_2, \$)\}$	Push
q_2			$\{(q_2, 0)\}$ 1	$\{(q_3, \varepsilon)\}$ 2				4	5	
q_3				$\{(q_3, \varepsilon)\}$ 3				$\{(q_4, \varepsilon)\}$		
q_4										



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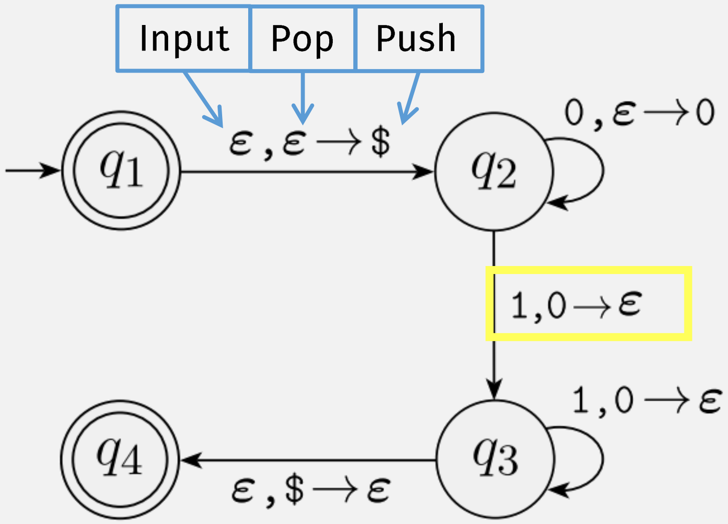
Input:	0			1			ε		
Stack:	0	\$	ε	0	\$	ε	0	\$	ε
q_1									$\{(q_2, \$)\}$
q_2			$\{(q_2, 0)\}$	$\{(q_3, \varepsilon)\}$					
q_3				$\{(q_3, \varepsilon)\}$				$\{(q_4, \varepsilon)\}$	
q_4									



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δ is given by the following table, wherein blank entries signify $\emptyset$.

Input:	0			1			ε			Input
Stack:	0	\$	ε	0	\$	ε	0	\$	ε	Pop
q_1									$\{(q_2, \$)\}$	Push
q_2			$\{(q_2, 0)\}$	$\{(q_3, \varepsilon)\}$ 2				4	5	
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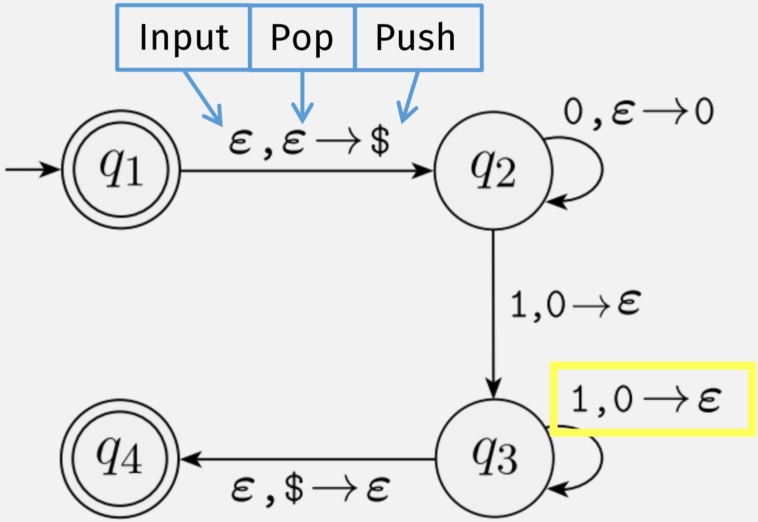

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Input Pop Push

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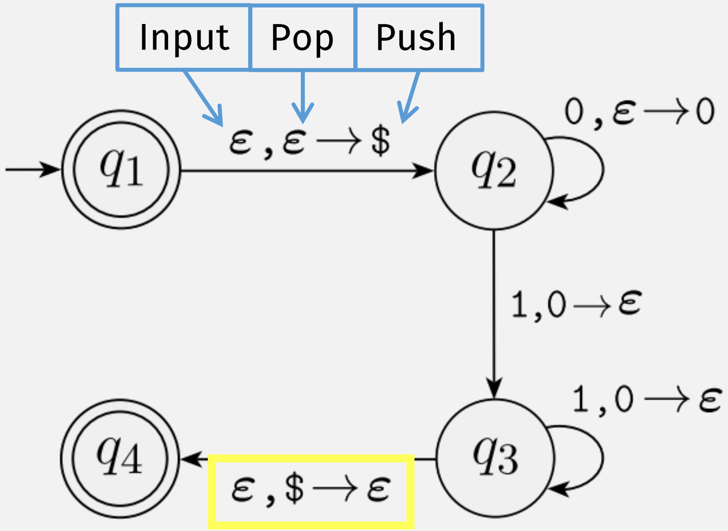
$F = \{q_1, q_4\}$, and

δ is given by the following table, wherein blank entries signify $\emptyset$.

Input:	0			1			ε (Input)		
Stack:	0	\$	ε	0	\$	ε	0	\$	ε (Pop)
q_1									$\{(q_2, \$)\}$ (Push) (5)
q_2			$\{(q_2, 0)\}$ (1)	$\{(q_3, \varepsilon)\}$ (2)					
q_3				$\{(q_3, \varepsilon)\}$ (3)				$\{(q_4, \varepsilon)\}$ (4)	
q_4									



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Input Pop Push

Flashback: DFA Computation Model

Informally	*Formally (i.e., mathematically)*

- Computer = a DFA
- Program = input string of chars, e.g. "`1101`"

To run a program:

- Start in "start state"
- Read 1 char at a time, changing states according to the transition table
- Result =
  - "Accept" if last state is "Accept" state
  - "Reject" otherwise

- $M = (Q, \Sigma, \delta, q_0, F)$
- $w = w_1 w_2 \cdots w_n$
- $r_0 = q_0$

For DFA, a single state represents a "**snapshot**" of the computation

- $\delta(r_i, w_{i+1}) = r_{i+1}, \text{ for } i = 0, \ldots, n-1$

Sequence of states **completely** represents a computation

- M ***accepts*** w if

sequence of states $r_0, r_1, \ldots, r_n$ in Q exists …

with $r_n \in F$

PDA Configurations (IDs)

- A **configuration** (or **ID**) is a "snapshot" of a PDA's computation

- A configuration (or **ID**) (q, w, γ) has three components:
  q = the current state
  w = the remaining input string
  γ = the stack contents

- A sequence of configurations represents a PDA computation

PDA Computation, Formally

$$P = (Q, \Sigma, \Gamma, \delta, q_0, F)$$

Single-step

Before / After configurations

$$(q_1, aw, X\beta) \vdash (q_2, w, \alpha\beta)$$

Read Input | Pop | Push

$$\text{if } \delta(q_1, a, X) \text{ contains } (q_2, \alpha)$$

$$q_1, q_2 \in Q$$

$$a \in \Sigma \quad w \in \Sigma^*$$

$$X \in \Gamma \quad \beta, \alpha \in \Gamma^*$$

A configuration (q, w, γ) has three components

- q = the current state
- w = the remaining input string
- γ = the stack contents

Extended

- Base Case

$$I \overset{*}{\vdash} I \text{ for any ID } I$$

- Recursive Case

$$I \overset{*}{\vdash} J \text{ if there exists some ID } K$$
$$\text{such that } I \vdash K \text{ and } K \overset{*}{\vdash} J$$

Language of a PDA

$$P = (Q, \Sigma, \Gamma, \delta, q_0, F)$$

Start in start state

Stack initially empty

Computation ends when input is completely read

$$L(P) = \{w \mid (q_0, w, \varepsilon) \vdash^* (q, \varepsilon, \alpha)\} \text{ where } q \in F$$

Machine accepts if final state is accept state

A configuration (q, w, γ) has three components

q = the current state

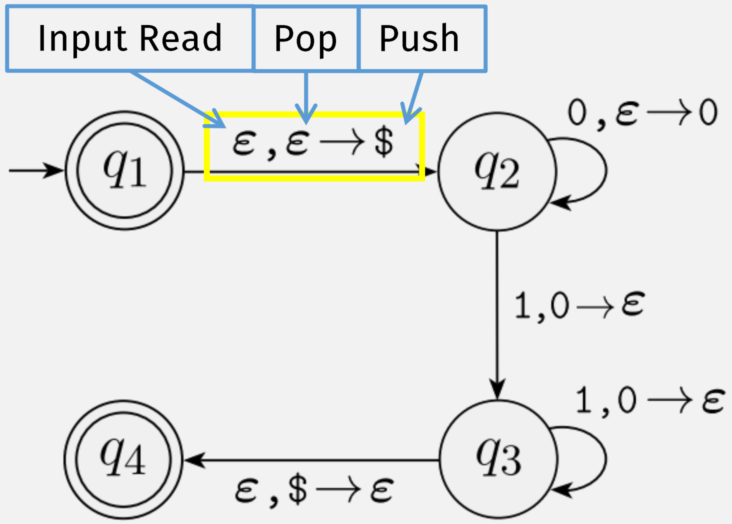
w = the remaining input string

γ = the stack contents

PDA Running Input String Example

State | Remaining Input | Stack

$$(q_1, 0011, \varepsilon)$$

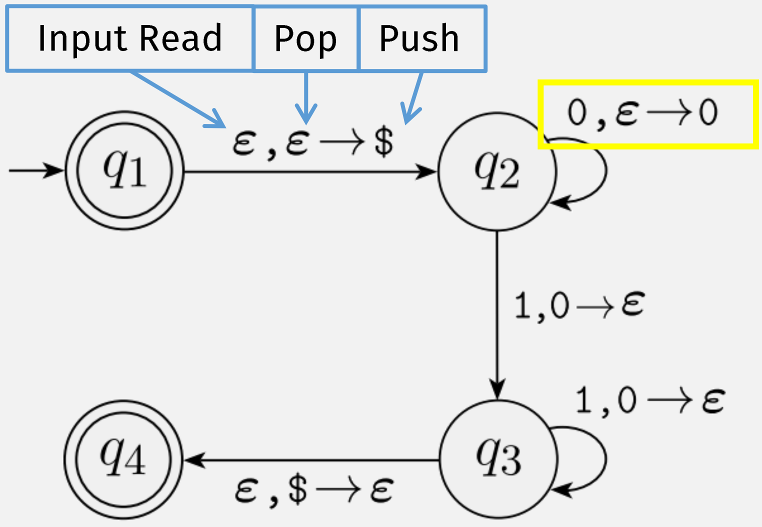


PDA Running Input String Example

State | Remaining Input | Stack

$$(q_1, 0011, \varepsilon) \vdash (q_2, 0011, \$)$$

$$\vdash (q_2, 011, 0\$)$$



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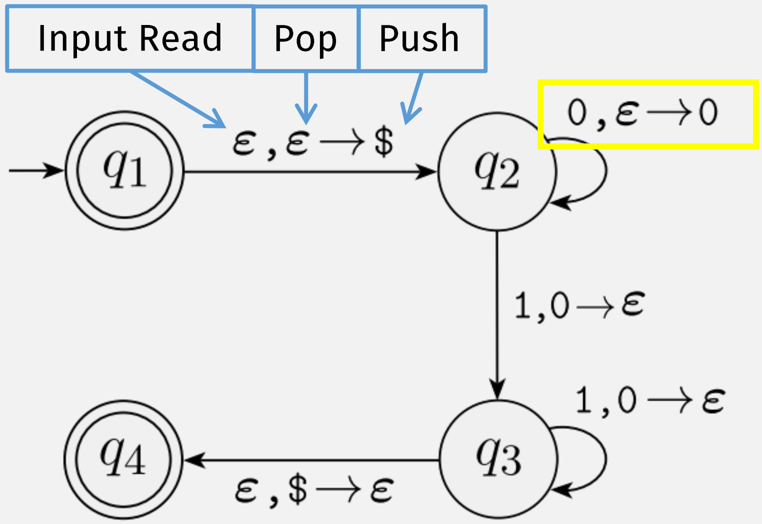


State | Remaining Input | Stack

$$(q_1, 0011, \varepsilon) \vdash (q_2, 0011, \$)$$

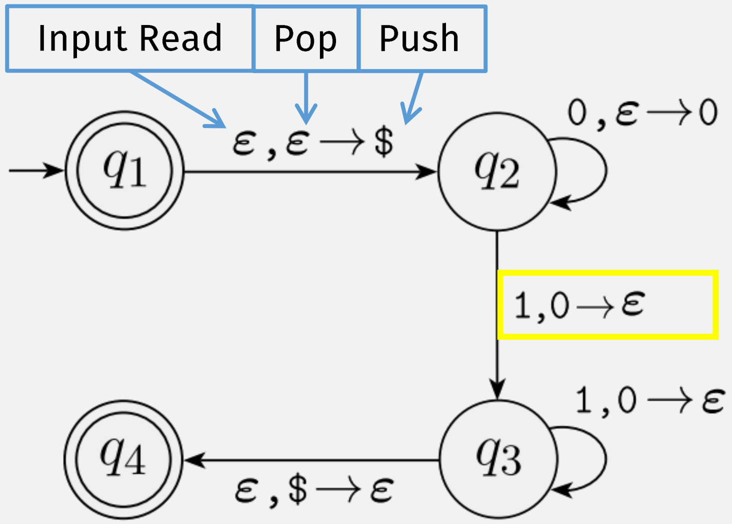
$$\vdash (q_2, 011, 0\$)$$

$$\vdash (q_2, 11, 00\$)$$

PDA Running Input String Example

State | Remaining Input | Stack

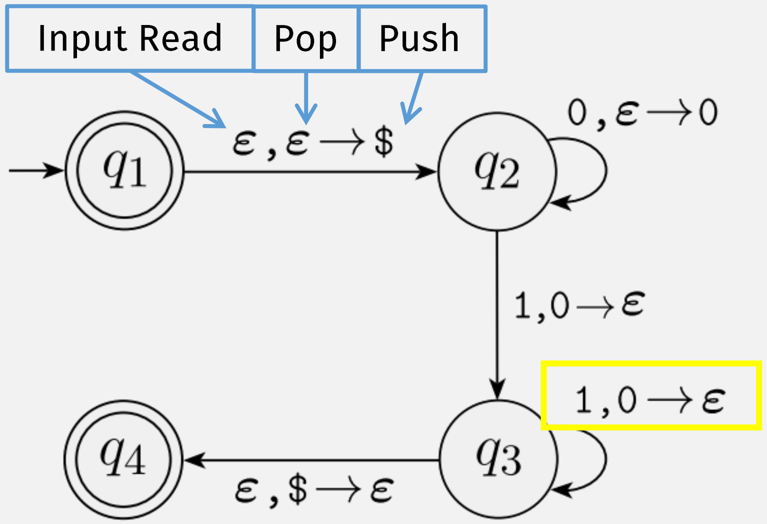
$$(q_1, 0011, \varepsilon) \vdash (q_2, 0011, \$)$$
$$\vdash (q_2, 011, 0\$)$$
$$\vdash (q_2, 11, 00\$)$$
$$\vdash (q_3, 1, 0\$)$$



PDA Running Input String Example

State | Remaining Input | Stack

$$
\begin{aligned}
(q_1, 0011, \varepsilon) &\vdash (q_2, 0011, \$) \\
&\vdash (q_2, 011, 0\$) \\
&\vdash (q_2, 11, 00\$) \\
&\vdash (q_3, 1, 0\$) \\
&\vdash (q_3, \varepsilon, \$)
\end{aligned}
$$



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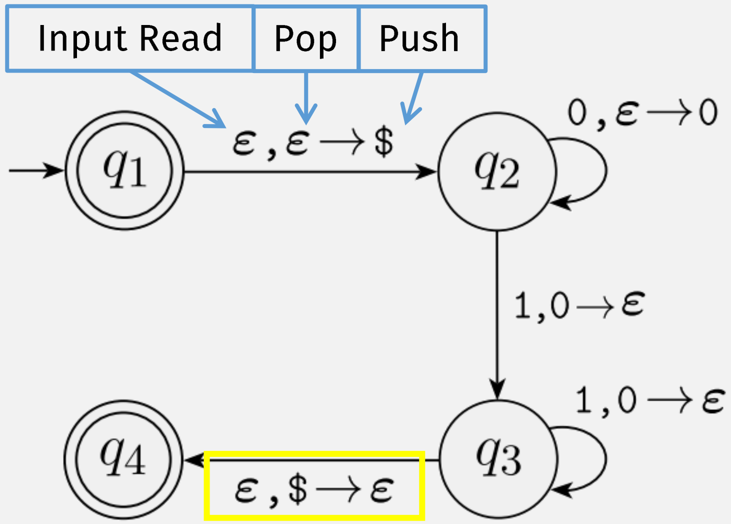
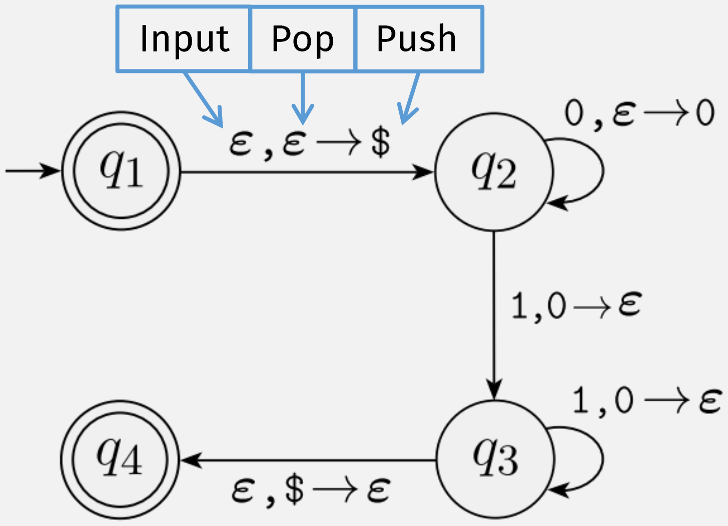


State | Remaining Input | Stack

$$
\begin{aligned}
(q_1, 0011, \varepsilon) &\vdash (q_2, 0011, \$) \\
&\vdash (q_2, 011, 0\$) \\
&\vdash (q_2, 11, 00\$) \\
&\vdash (q_3, 1, 0\$) \\
&\vdash (q_3, \varepsilon, \$) \\
&\vdash (q_4, \varepsilon, \varepsilon)
\end{aligned}
$$

Pushdown Automata (PDA)



- PDA = NFA + a stack
  - Infinite memory
  - Can only read/write top location: Push/pop
- Want to prove: **PDAs represent CFLs!**
- We know: **a CFL, by definition, is a language that is generated by a CFG**
- Need to show: **PDA ⇔ CFG**
- **Then, to prove that a language is a CFL, we can either:**
  - **Create a CFG**, or
  - **Create a PDA**

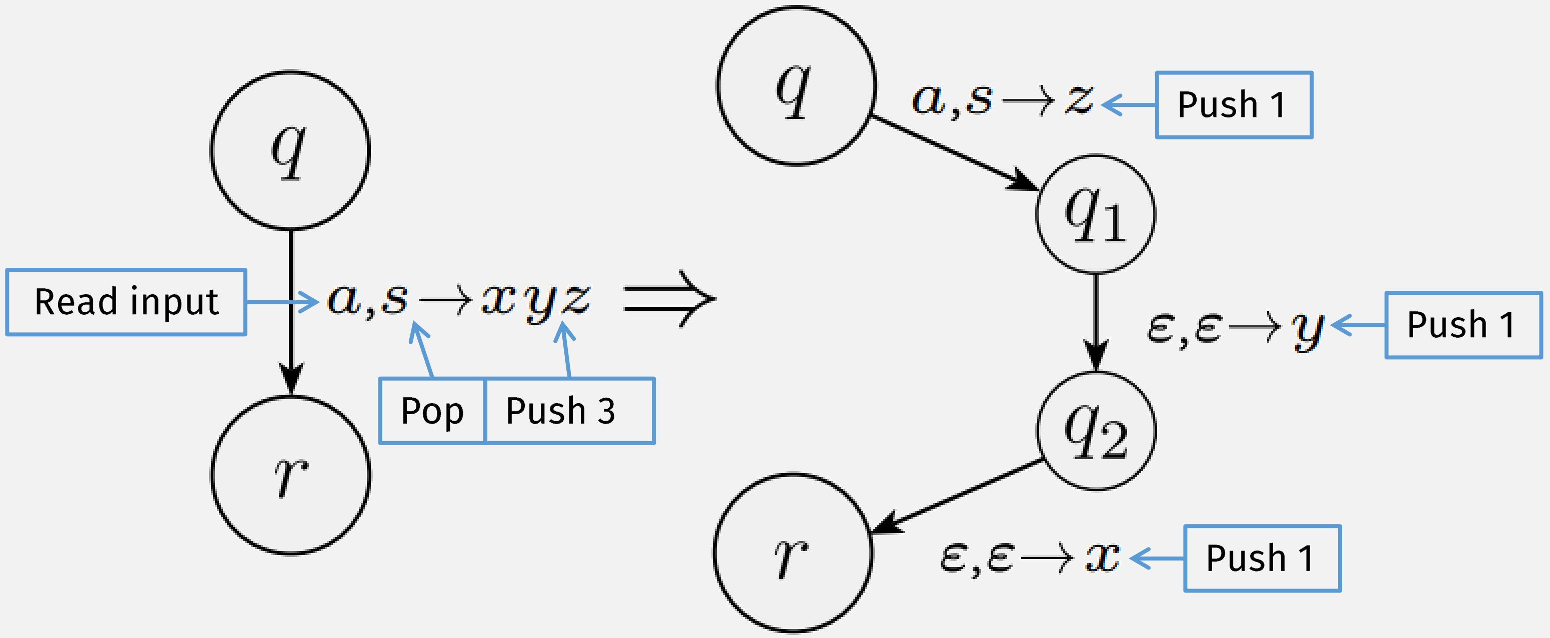
A lang is a CFL **iff** some PDA recognizes it

$\Rightarrow$ If a language is a CFL, then a PDA recognizes it

- (Easier)
- We know: **A CFL has a CFG describing it** (definition of CFL)
- Must show: **the CFG has an equivalent PDA**

$\Leftarrow$ If a PDA recognizes a language, then it's a CFL

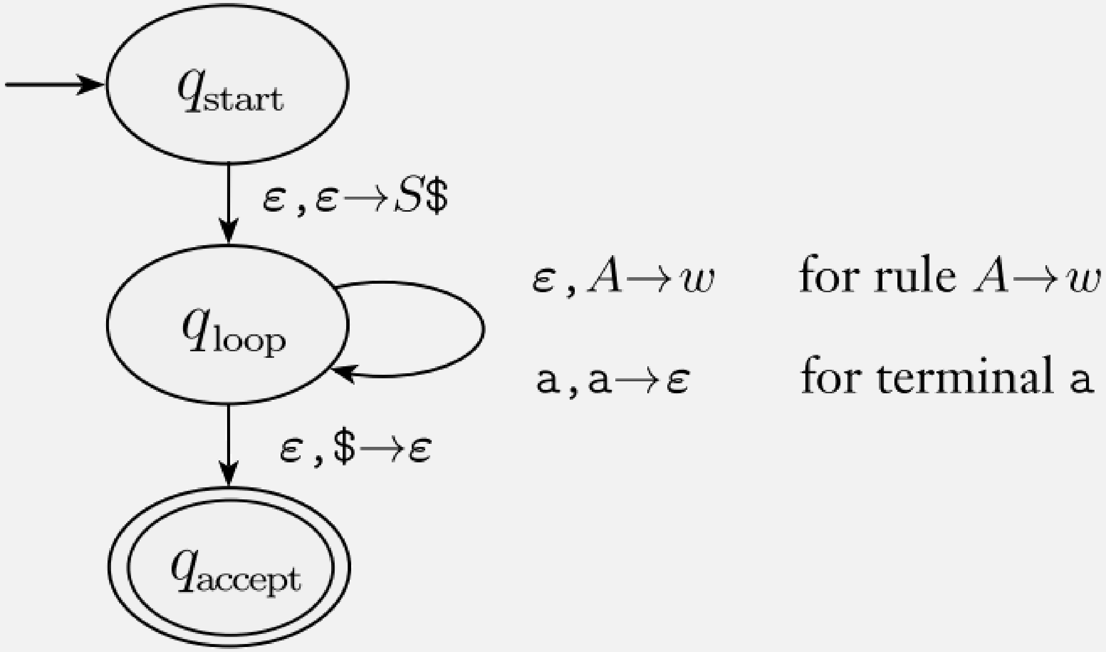
Shorthand: Multi-Symbol Stack Pushes



Note the <u>reverse</u> order of pushes

CFG→PDA (sketch)

- Construct a PDA from CFG such that:
  - PDA accepts input string only if the CFG can generate that string
- Intuitively, PDA will nondeterministically try all rules



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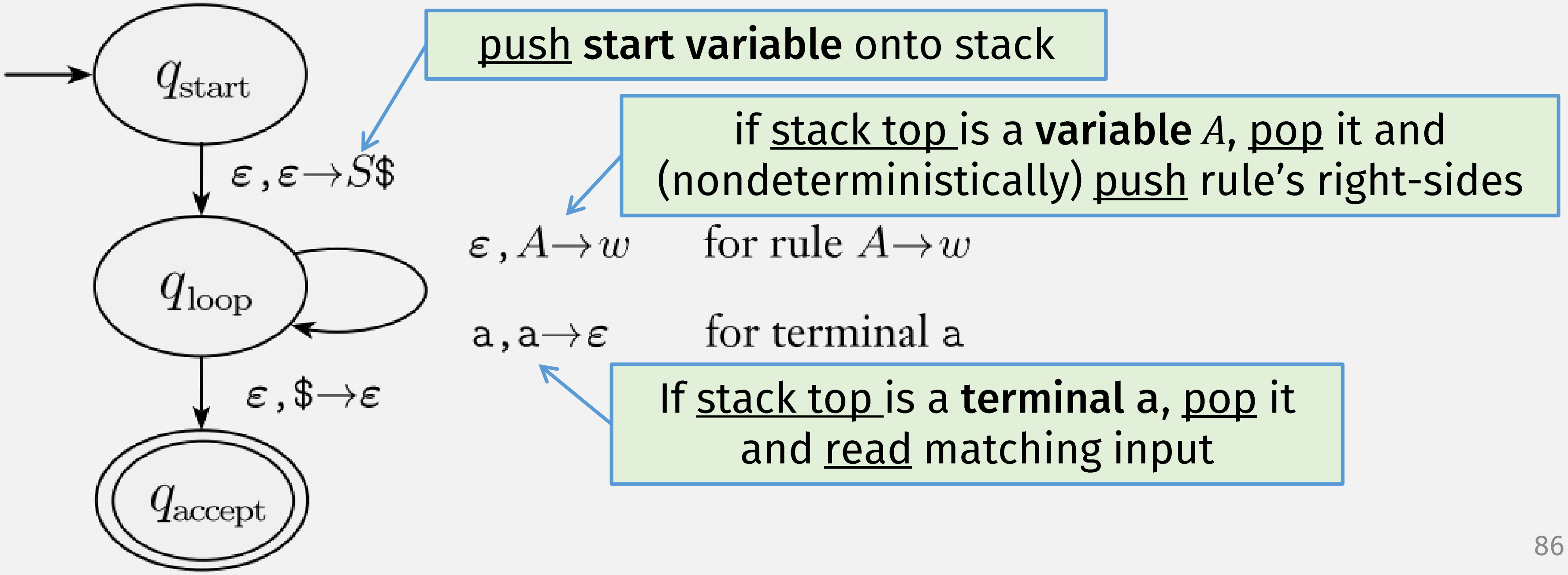
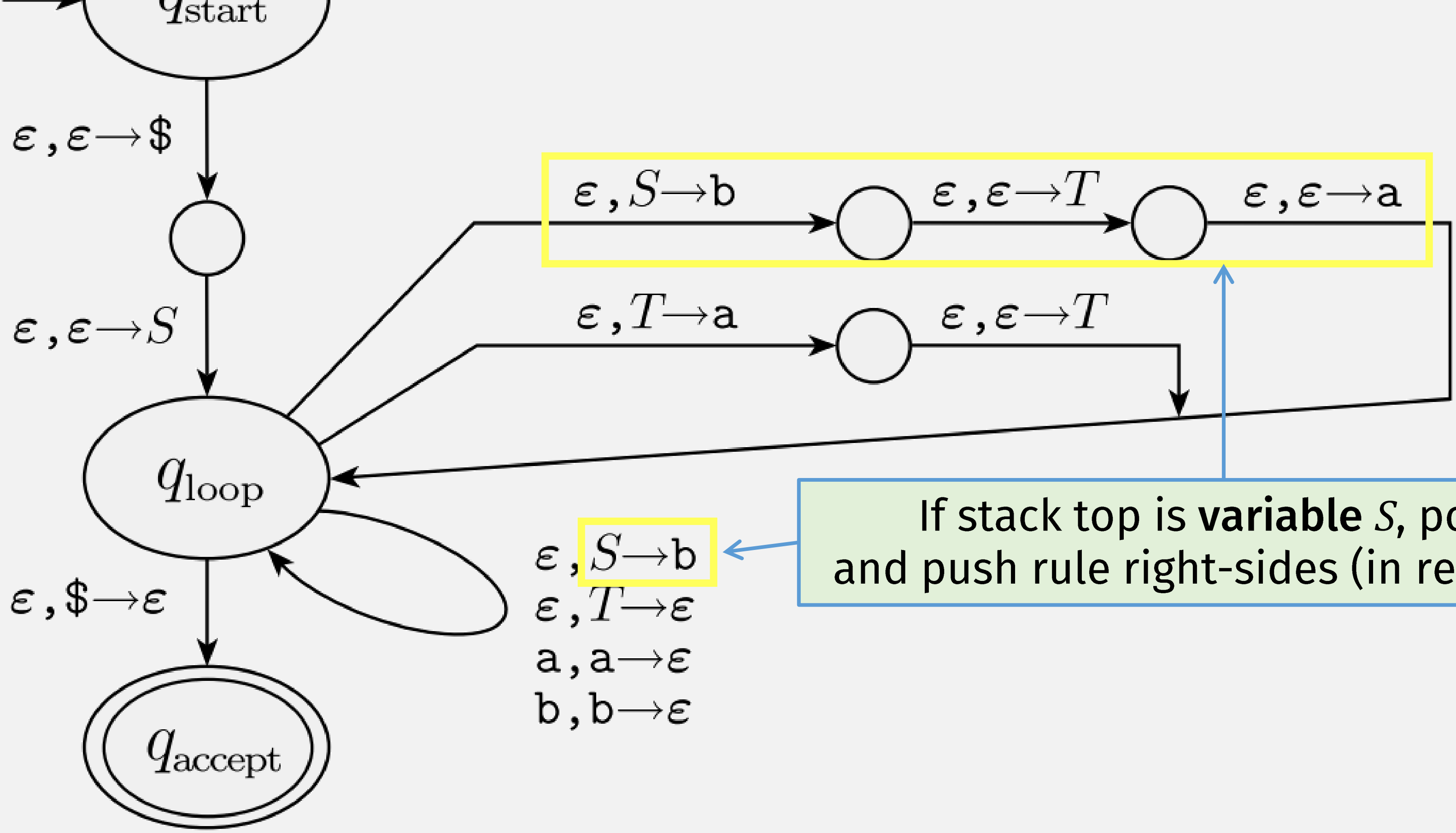


Example **CFG→PDA**

$$S \to \texttt{a}T\texttt{b} \mid \texttt{b}$$
$$T \to T\texttt{a} \mid \varepsilon$$



If stack top is **variable** S, pop S
and push rule right-sides (in rev order)

Example **CFG→PDA**

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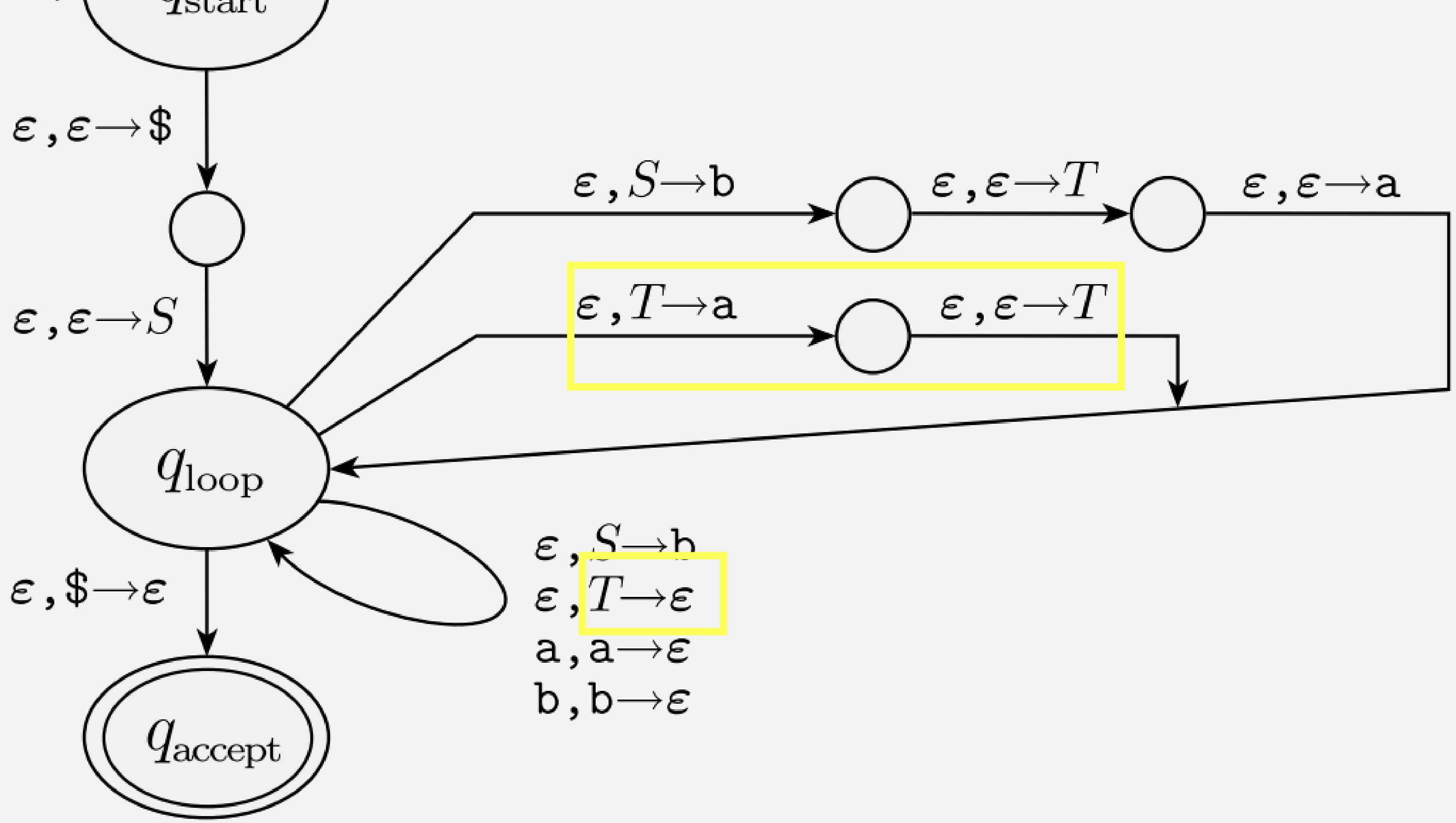

Example CFG→PDA

$$S \to \mathtt{a}T\mathtt{b} \mid \mathtt{b}$$
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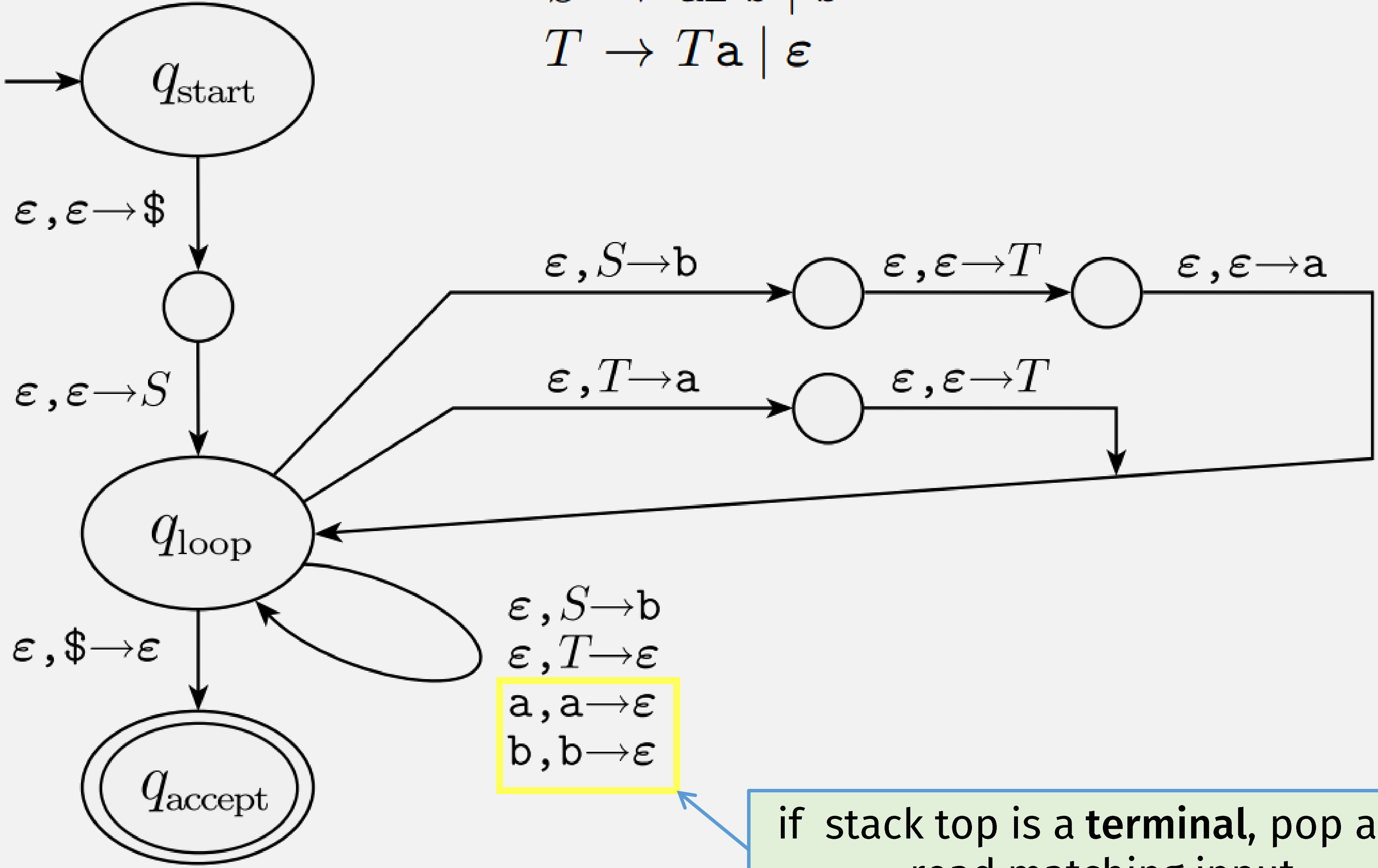


if stack top is a **terminal**, pop and read matching input

Example CFG→PDA

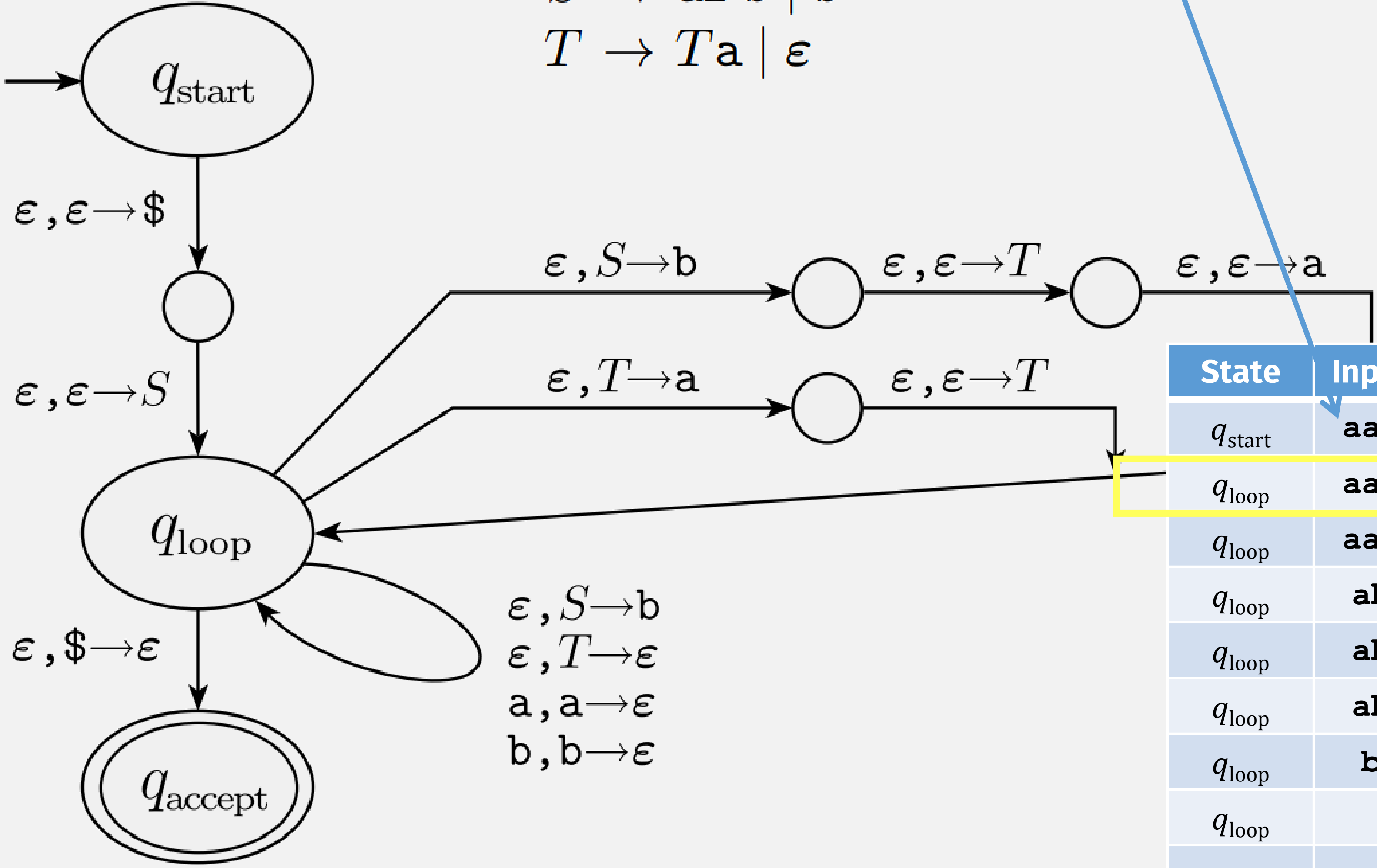
Example Derivation using CFG:

$S \Rightarrow \mathtt{a}T\mathtt{b}$ (using rule $S \to \mathtt{a}T\mathtt{b}$)

$\Rightarrow \mathtt{a}T\mathtt{ab}$ (using rule $T \to T\mathtt{a}$)

$\Rightarrow \mathtt{aab}$ (using rule $T \to \varepsilon$)

$$S \to \mathtt{a}T\mathtt{b} \mid \mathtt{b}$$
$$T \to T\mathtt{a} \mid \varepsilon$$



PDA Example

State	Input	Stack	Equiv Rule
q_{start}	aab		
q_{loop}	aab	S\$	
q_{loop}	aab	aTb\$	$S \to \mathtt{a}T\mathtt{b}$
q_{loop}	ab	Tb\$	
q_{loop}	ab	Tab\$	$T \to T\mathtt{a}$
q_{loop}	ab	ab\$	$T \to \varepsilon$
q_{loop}	b	b\$	
q_{loop}		\$	
q_{accept}			

Example **CFG→PDA**

Example Derivation using CFG:

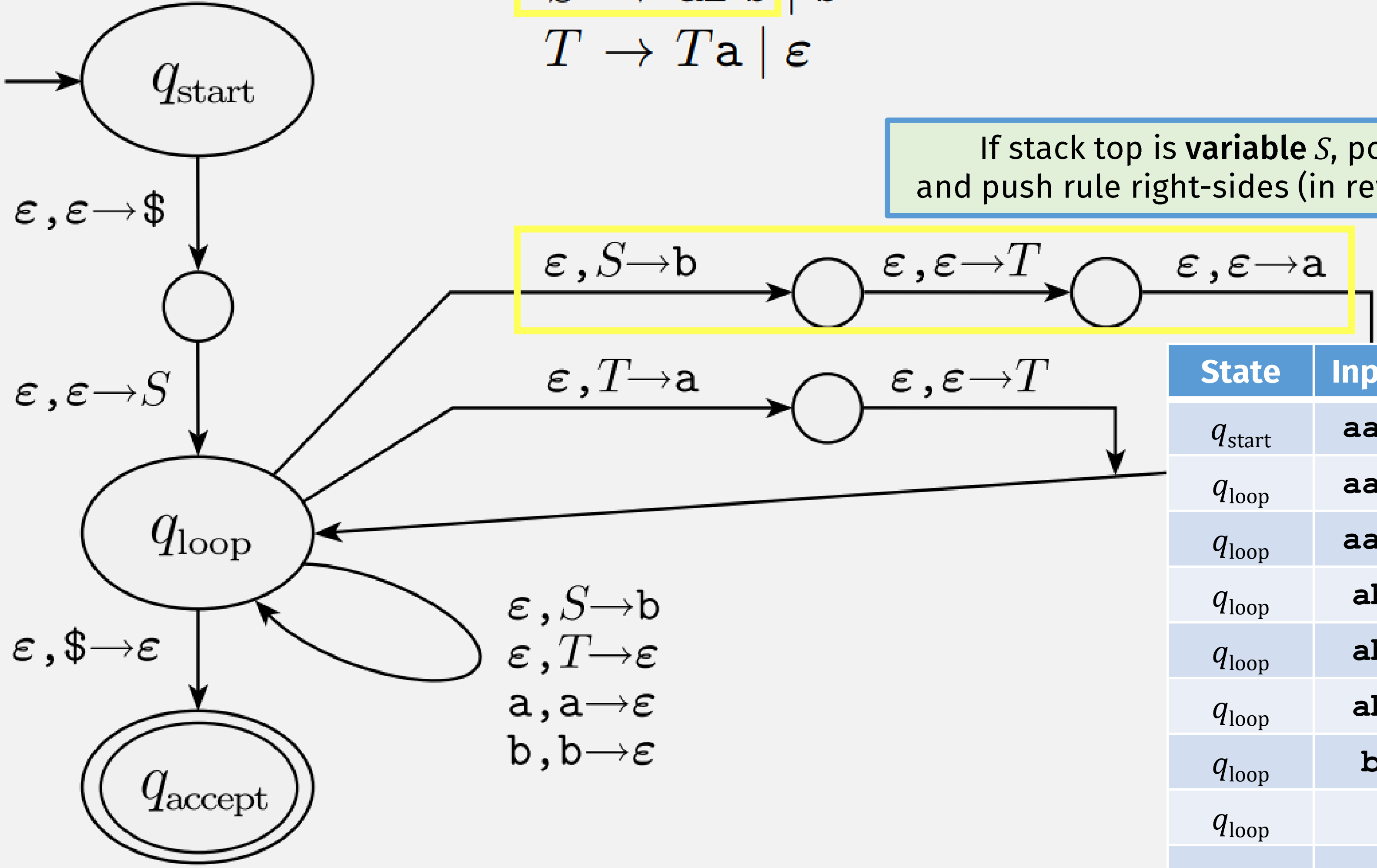
$S \Rightarrow \mathtt{a}T\mathtt{b}$ (using rule $S \rightarrow \mathtt{a}T\mathtt{b}$)

$\Rightarrow \mathtt{a}T\mathtt{ab}$ (using rule $T \rightarrow T\mathtt{a}$)

$\Rightarrow \mathtt{aab}$ (using rule $T \rightarrow \varepsilon$)

$$S \rightarrow \mathtt{a}T\mathtt{b} \mid \mathtt{b}$$
$$T \rightarrow T\mathtt{a} \mid \varepsilon$$

If stack top is **variable** S, pop S and push rule right-sides (in rev order)



PDA Example

State	Input	Stack	Equiv Rule
q_{start}	aab		
q_{loop}	aab	$S\$$	
q_{loop}	aab	$\mathtt{a}T\mathtt{b}\$$	$S \rightarrow \mathtt{a}T\mathtt{b}$
q_{loop}	ab	$T\mathtt{b}\$$	
q_{loop}	ab	$T\mathtt{ab}\$$	$T \rightarrow T\mathtt{a}$
q_{loop}	ab	ab$	$T \rightarrow \varepsilon$
q_{loop}	b	b$	
q_{loop}		$	
q_{accept}			

Example **CFG→PDA**

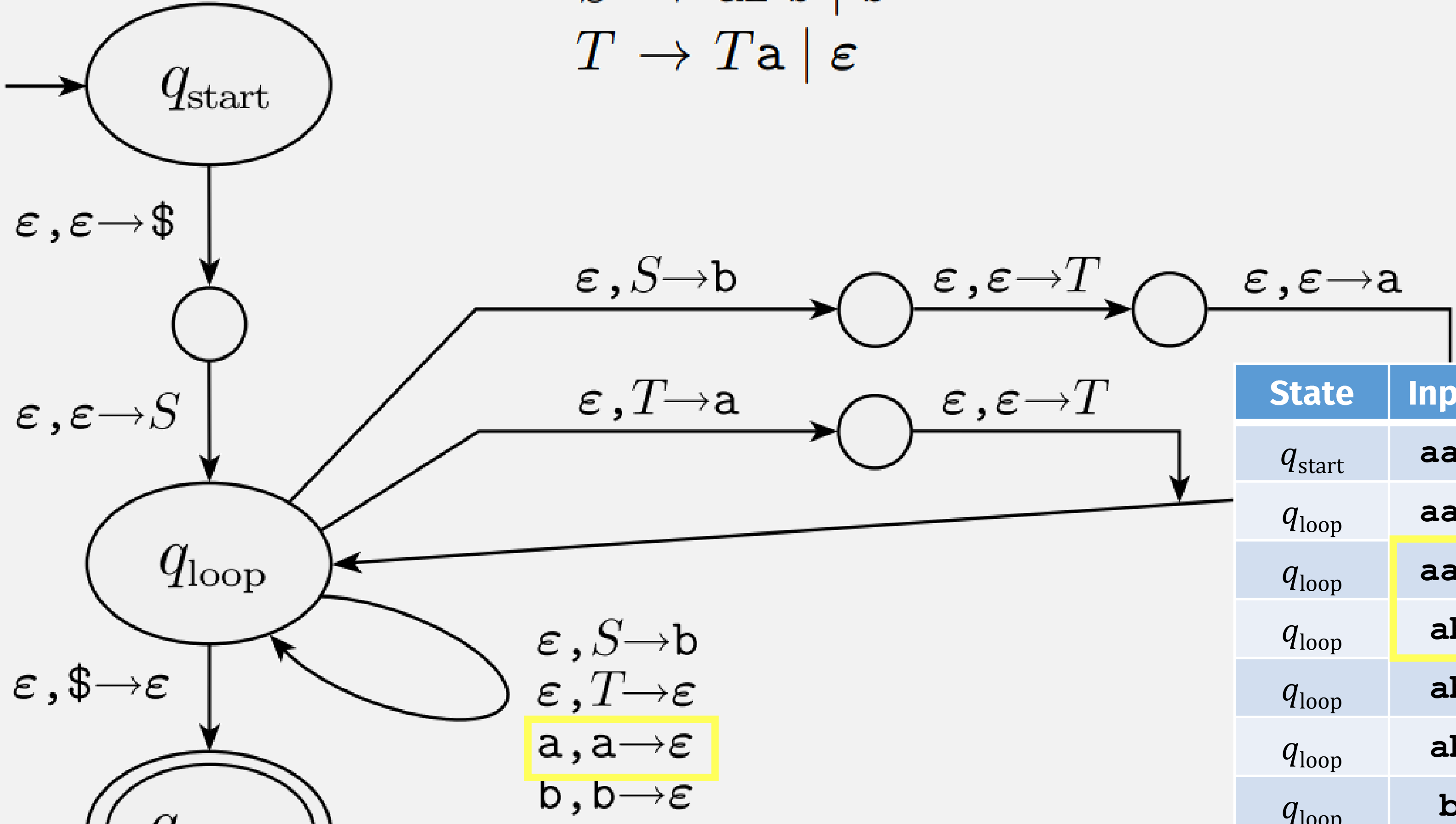
Example Derivation using CFG:

$S \Rightarrow \mathbf{a}T\mathbf{b}$ (using rule $S \to \mathbf{a}T\mathbf{b}$)

$\Rightarrow \mathbf{a}T\mathbf{ab}$ (using rule $T \to T\mathbf{a}$)

$\Rightarrow \mathbf{aab}$ (using rule $T \to \varepsilon$)

$$S \to \mathtt{a}T\mathtt{b} \mid \mathtt{b}$$
$$T \to T\mathtt{a} \mid \varepsilon$$



if stack top is a **terminal**, pop and read matching input

PDA Example

State	Input	Stack	Equiv Rule
q_{start}	**aab**		
q_{loop}	**aab**	$S\$$	
q_{loop}	**aab**	$\mathbf{a}T\mathbf{b}\$$	$S \to \mathbf{a}T\mathbf{b}$
q_{loop}	**ab**	$T\mathbf{b}\$$	
q_{loop}	**ab**	$T\mathbf{ab}\$$	$T \to T\mathbf{a}$
q_{loop}	**ab**	$\mathbf{ab}\$$	$T \to \varepsilon$
q_{loop}	**b**	$\mathbf{b}\$$	
q_{loop}		$\$$	
q_{accept}			

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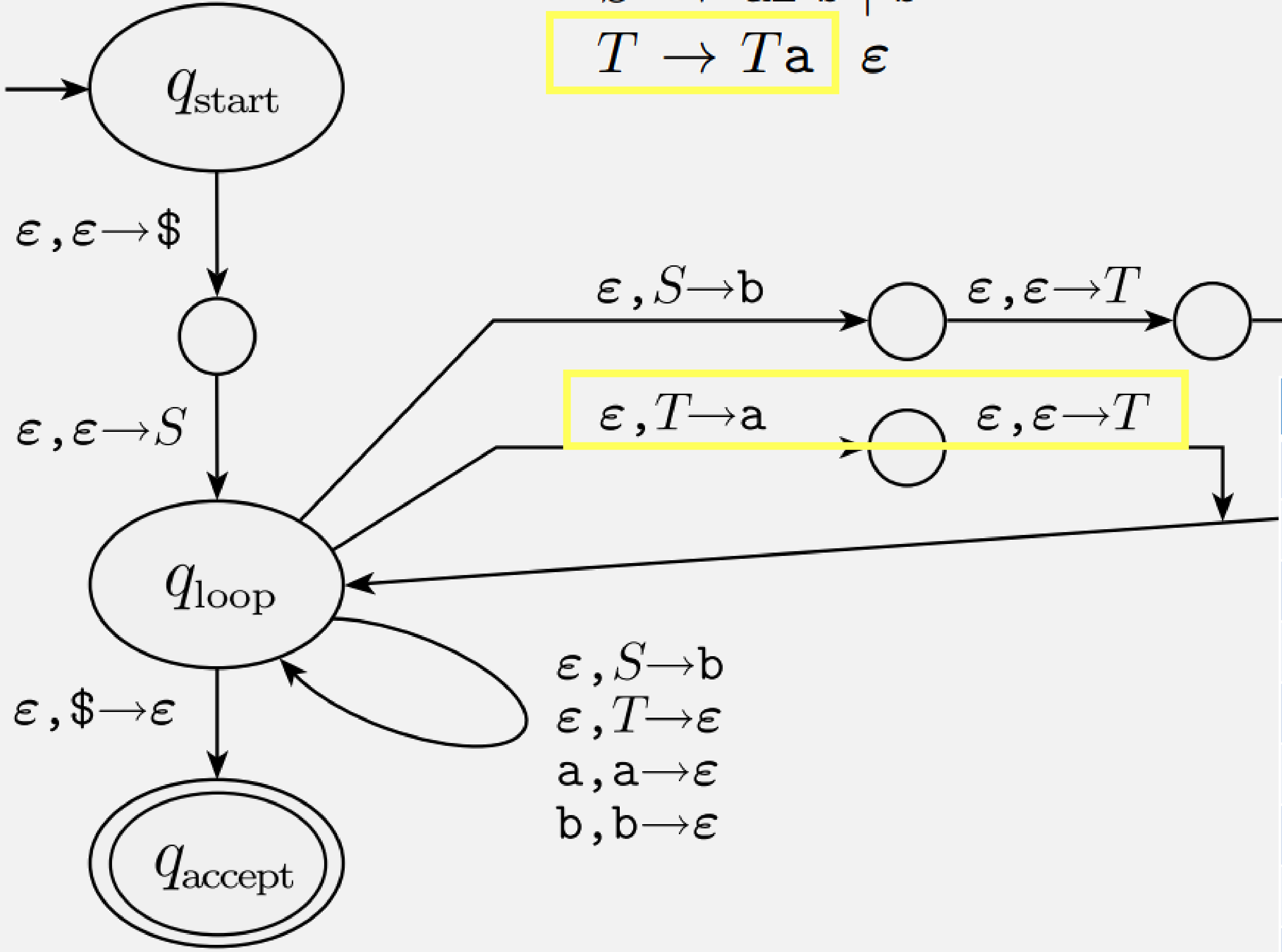

PDA Example

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q_{loop}	aab	$\mathtt{a}T\mathtt{b}\$$	$S \rightarrow \mathtt{a}T\mathtt{b}$
q_{loop}	ab	$T\mathtt{b}\$$	
q_{loop}	ab	$T\mathtt{ab}\$$	$T \rightarrow T\mathtt{a}$
q_{loop}	ab	ab$	$T \rightarrow \varepsilon$
q_{loop}	b	b$	
q_{loop}		$	
q_{accept}			

A lang is a CFL **iff** some PDA recognizes it

☑ ⇒ If a language is a CFL, then a PDA recognizes it

- Convert CFG→PDA

⇐ If a PDA recognizes a language, then it's a CFL

- (Harder)
- Must Show: PDA has an equivalent CFG

PDA→CFG: Prelims

Before converting PDA to CFG, modify it so :

1. It has a single accept state, q_{accept}.
2. It empties its stack before accepting.
3. Each transition either pushes a symbol onto the stack (a *push* move) or pops one off the stack (a *pop* move), but it does not do both at the same time.

Important:
This doesn't change the language recognized by the PDA

PDA P -> CFG G: Variables

$P = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{\text{accept}}\})$ variables of G are $\{A_{pq} \mid p, q \in Q\}$

- Want: if P goes from state p to q reading input x, then some A_{pq} generates x
- So: For every pair of states p, q in P, add variable A_{pq} to G
- Then: connect the variables together by,
  - Add rules: $A_{pq} \rightarrow A_{pr}A_{rq}$, for each state r
  - These rules allow grammar to simulate every possible transition
  - (We haven't added input read/generated terminals yet)
- To add terminals: pair up stack pushes and pops (essence of a CFL)

PDA P -> CFG G: Generating Strings

$P = (Q, \Sigma, \Gamma, \delta, q_0, \{q_{\text{accept}}\})$

variables of G are $\{A_{pq} \mid p, q \in Q\}$

- The key: **pair up stack pushes and pops** (essence of a CFL)

if $\delta(p, a, \varepsilon)$ contains (r, u) and $\delta(s, b, u)$ contains (q, ε),

put the rule $A_{pq} \to aA_{rs}b$ in G

PDA P -> CFG G: Generating Strings

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A language is a CFL ⇔ A PDA recognizes it

☑ ⇒ If a language is a CFL, then a PDA recognizes it

- Convert **CFG→PDA**

☑ ⇐ If a PDA recognizes a language, then it's a CFL

- Convert **PDA→CFG**

∎

Check-in Quiz 2/23

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