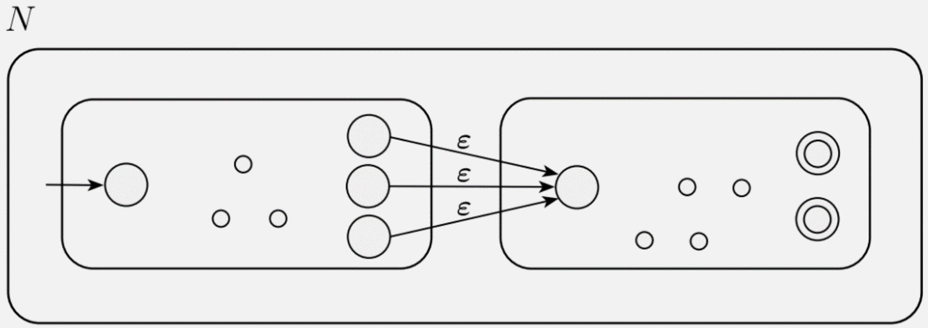
CS420

Combining Automata & Closed Operations

Monday, February 6, 2023

UMass Boston Computer Science



Announcements

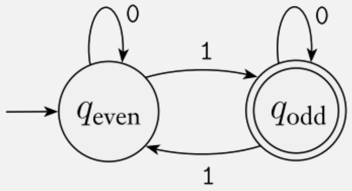
- HW 1
  - Due Tue 2/7 11:59pm EST

Quiz Preview

- To prove the statement:
  - “The set of regular languages is **closed** under the union operation”
- What is the equivalent IF-THEN statement to prove?

Last Time: Proving a Language is Regular

Statements

1. If an FSM recognizes L, then L is a regular language
2. M = 

is an FSM

Key is creating this FSM!

3. M recognizes L
4. $L = \{ w \mid w$ is string with odd # of `1`s$\}$ is a regular language

Justifications

1. Def. of a Regular Language
2. Definition of an FSM
3. See examples. This isn't a proof, but good enough for programmers(?), and CS 420
4. Stmt #1 & #3 (modus ponens)

Last Time: Tips on Designing Finite Automata

Analogy
Finite Automata ~ "Programs" ::
Designing Finite Automata ~ "Programming"!

1. Confirm understanding of the problem
   - Create tests: example inputs vs expected results (`accept` / `reject`)

2. Decide information that machine "remembers"
   - These are the machine states: some are accept states; one is start state

3. Determine transitions between states

4. Test machine behaves as expected
   - Use initial examples; and create additional tests if needed

Last Time: Combining DFAs?

Password Requirements

- Passwords must have a minimum length of ten (10) characters - but more is better!
- Passwords **must include at least 3** different types of characters:
  - upper-case letters (A-Z)
  - lower-case letters (a-z)
  - symbols or special characters (%, &, *, $, etc.)
  - numbers (0-9)
- Passwords cannot contain all or part of your email address
- Passwords cannot be re-used

DFA

DFA

DFA

DFA

DFA

DFA

DFA

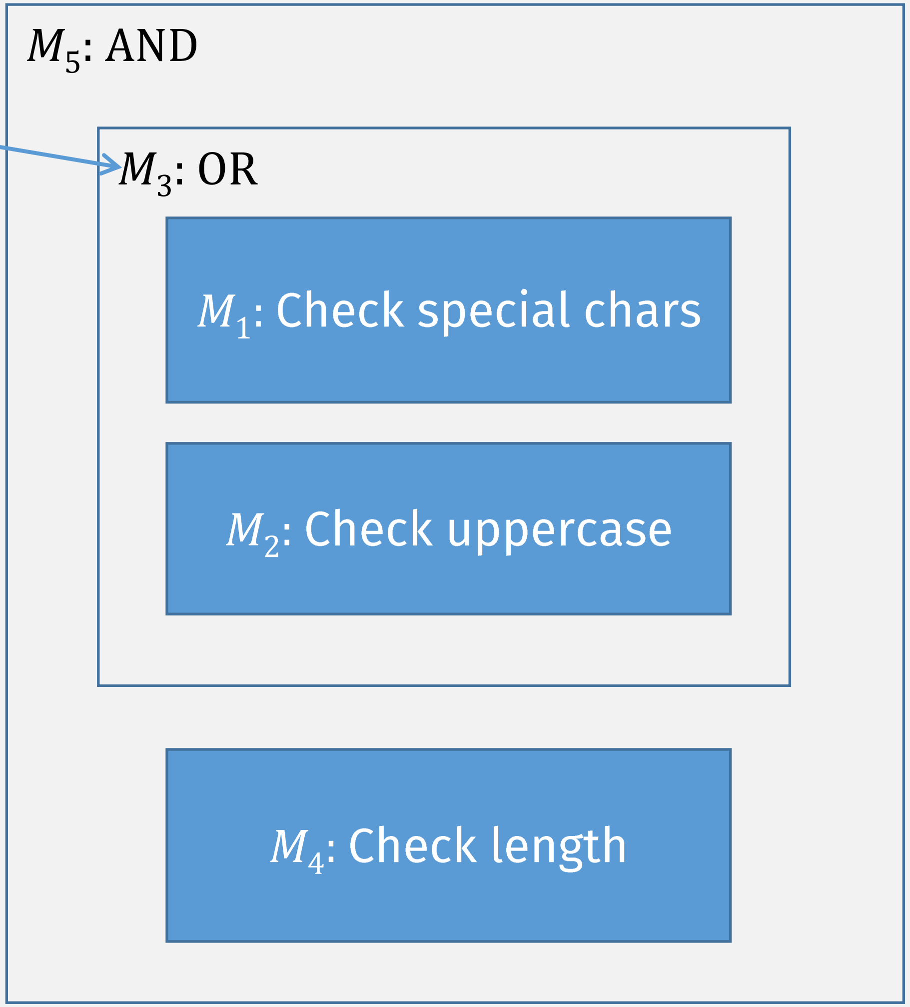
https://www.umb.edu/it/password

To match all requirements, combine smaller DFAs into one big DFA?

(We do this with programs all the time)

Password Checker DFAs

What if this is not a DFA?



Want to be able to easily <u>combine</u> DFAs, i.e., <u>composability</u>

<u>We want these operations:</u>

$\text{OR} : \text{DFA} \times \text{DFA} \rightarrow \text{DFA}$

$\text{AND} : \text{DFA} \times \text{DFA} \rightarrow \text{DFA}$

To <u>combine more than once</u>, operations must be **closed**!

“Closed” Operations

- Set of Natural numbers = $\{0, 1, 2, \ldots\}$
  - Closed under addition:
    - if x and y are Natural numbers,
    - then $z = x + y$ is a Natural number
  - Closed under multiplication?
    - **yes**
  - Closed under subtraction?
    - **no**
- Integers = $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
  - Closed under addition and multiplication
  - Closed under subtraction?
    - **yes**
  - Closed under division?
    - **no**
- Rational numbers = $\{x \mid x = y/z, y \text{ and } z \text{ are Integers}\}$
  - Closed under division?
    - **No?**
    - **Yes if** $z \mathrel{!=} 0$

A set is **closed** under an operation if: result of the operation is in the same set as inputs to the operation

We Want "Closed" Ops For Regular Langs!

- Set of Regular Languages = $\{L_1, L_2, \ldots\}$
  - Closed under …?
    - OR (union)
    - AND (intersection)
    - …

A set is **closed** under an operation if: result of the operation is in the same set as inputs to the operation

Why Care About Closed Ops on Reg Langs?

- Closed operations for regulars langs preserve "regularness"
- I.e., it preserves the same computation model!
- This allows "combining" smaller computation to get bigger ones:

<u>For Example:</u>

OR: Regular Lang × Regular Lang → Regular Lang

- So this semester, we will look for operations that are **closed**!

Union: $A \cup B = \{x | \, x \in A \text{ or } x \in B\}$

Union of Languages

Let the alphabet Σ be the standard 26 letters $\{\texttt{a}, \texttt{b}, \ldots, \texttt{z}\}$.

If $A = \{\texttt{good}, \texttt{bad}\}$ and $B = \{\texttt{boy}, \texttt{girl}\}$, then

$$A \cup B = \{\texttt{good}, \texttt{bad}, \texttt{boy}, \texttt{girl}\}$$

Is Union Closed For Regular Langs?

In this course, **we are interested in closed operations for a <u>set of languages</u>** (here the **set of regular languages**)

(*In general, a* **set** is **closed** under an operation if **applying the operation** to **members of the set** produces a **result in the same set**)

The class of regular languages is closed under the union operation.

Want to prove this statement

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

Or this (same) statement

Is Union Closed For Regular Langs?

(*In general*, a **set** is **closed** under an operation if applying the operation to members of the set produces a result in the same set)

THEOREM

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

Want to prove this statement

Or this (same) statement

A **member of the set of regular languages** is …

… **a regular language,** which itself is a set (of strings) …

… so the **operations** we're interested in are **set operations**

Is Union Closed For Regular Langs?

THEOREM

The class of regular languages is closed under the union operation.

In other words, $\boxed{\text{if } A_1 \text{ and } A_2 \text{ are regular languages, so is } A_1 \cup A_2}$.

Want to prove this statement

Or this (same) statement

Flashback: Mathematical Statements: **IF-THEN**

Using:

- If we know: $P \rightarrow Q$ is TRUE,
  what do we know about P and Q individually?
  - Either P is FALSE (not too useful, **can't prove anything about** Q), or
  - If P is TRUE, then Q is TRUE (**modus ponens**)

Proving:

p	q	$p \rightarrow q$
True	True	True
True	False	False
False	True	True
False	False	True

Flashback: Mathematical Statements: **IF-THEN**

THEOREM

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

t Q), or

Would have to prove there are no regular languages (impossible)

Proving:

- To prove: $P \rightarrow Q$ is TRUE:
  - Prove P is FALSE (usually hard or impossible)
  - Assume P is TRUE, then prove Q is TRUE

p	q	$p \rightarrow q$
True	True	True
True	False	False
False	True	True
False	False	True

Is Union Closed For Regular Langs?

Statements

Do we know anything about A_1 and A_2?

1. A_1 and A_2 are regular languages
2. A DFA $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognizes A_1
3. A DFA $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognizes A_2

???

4. Construct DFA $M = (Q, \Sigma, \delta, q_0, F)$ (todo)
5. M recognizes $A_1 \cup A_2$

How to create this? Don't know what A_1 and A_2 are!

6. $A_1 \cup A_2$ is a regular language
7. The class of regular languages is closed under the union operation.

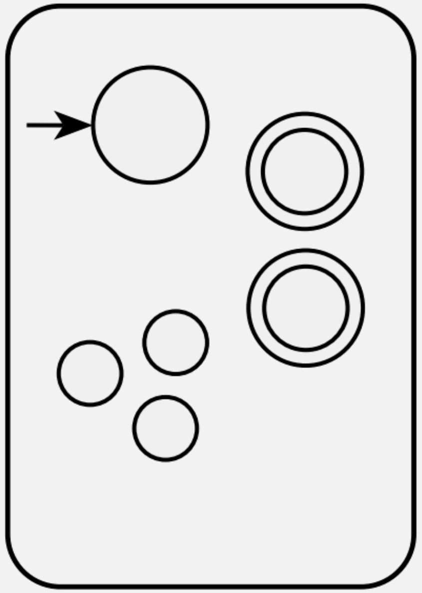
   In other words, $\boxed{\text{if } A_1 \text{ and } A_2 \text{ are regular languages, so is } A_1 \cup A_2}$.

Justifications

1. Assumption
2. Def of Regular Language
3. Def of Regular Language
4. Def of DFA
5. See examples
6. Def of Regular Language
7. From stmt #1 and #6

M_1

recognizes A_1



Regular language A_1

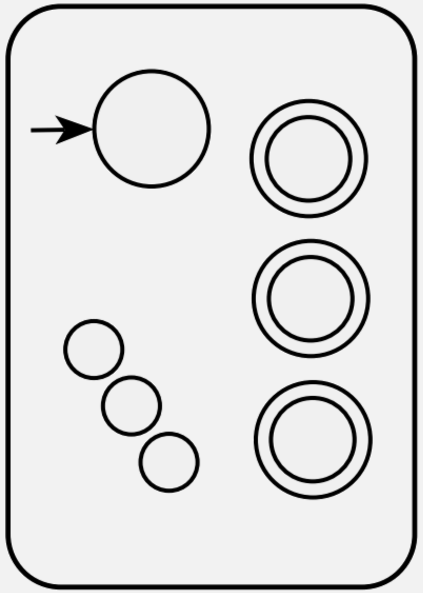
Regular language A_2

If we don't know what exactly these languages are, we still know these facts ...

A language is called a ***regular language*** if some finite automaton recognizes it.

M_2

recognizes A_2



DEFINITION

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

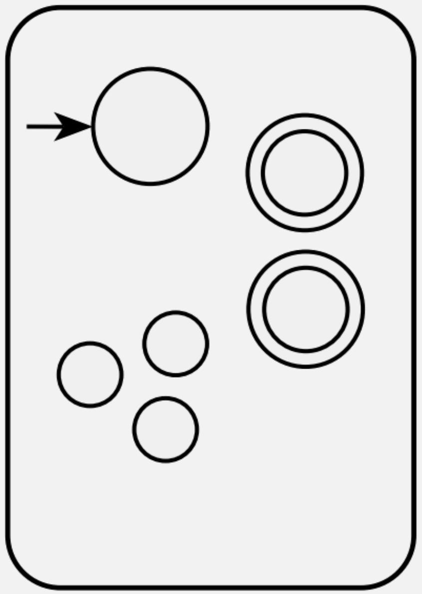
1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta: Q \times \Sigma \longrightarrow Q$ is the ***transition function***,
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

$$M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1), \text{ recognize } A_1,$$
$$M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2), \text{ recognize } A_2,$$

Union

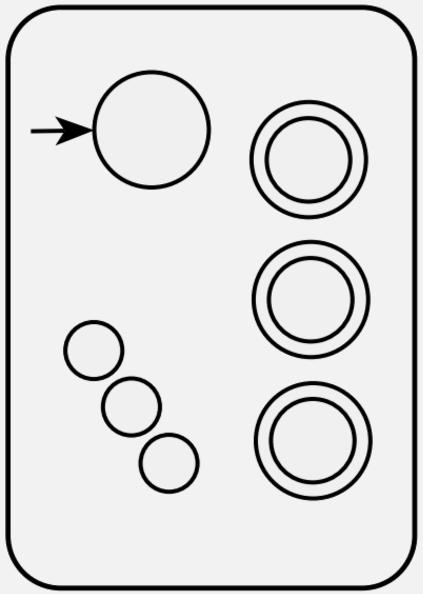
M_1

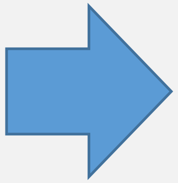
recognizes A_1



M_2

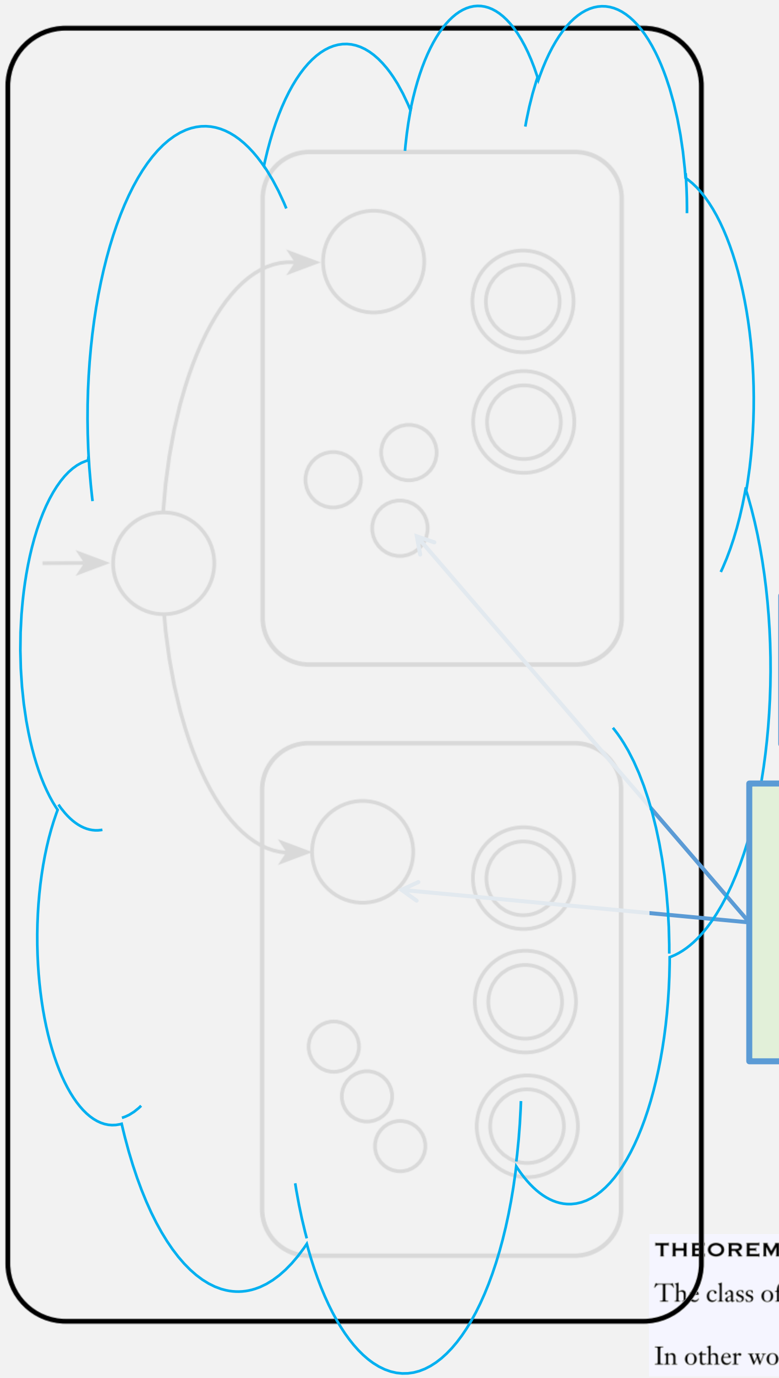
recognizes A_2





Want: M

Recognizes $A_1 \cup A_2$



Rough sketch Idea: M is a combination of M_1 and M_2 that checks whether its input is accepted by either M_1 and M_2

But, a DFA can only read its input once!

Need to somehow simulate "being in" both an M_1 and M_2 state simultaneously

THEOREM

The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

Union is Closed For Regular Languages

<u>Proof</u>

- Given: $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, recognize A_1,
  $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, recognize A_2,

Want: M that can simultaneously be in both an M_1 and M_2 state

- Construct: $M = (Q, \Sigma, \delta, q_0, F)$, using M_1 and M_2, that recognizes $A_1 \cup A_2$
- **states of M:** $Q = \{(r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$ $= Q_1 \times Q_2$
  This set is the ***Cartesian product*** of sets Q_1 and Q_2

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta \colon Q \times \Sigma \longrightarrow Q$ is the ***transition function***,[1]
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

A state of M is a <u>pair</u>:
- the <u>first</u> part is a state of M_1 and
- the <u>second</u> part is a state of M_2

So the states of M is <u>all possible combinations</u> of the states of M_1 and M_2

Union is Closed For Regular Languages

<u>*Proof*</u>

- Given: $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, recognize A_1,
  $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, recognize A_2,
- Construct: $M = (Q, \Sigma, \delta, q_0, F)$, using M_1 and M_2, that recognizes $A_1 \cup A_2$
- states of M: $Q = \{(r_1, r_2) | \, r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$ $= Q_1 \times Q_2$
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$a) = (\delta_1(r_1, a), \delta_2(r_2, a))$

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3. $\boxed{\delta \colon Q \times \Sigma \longrightarrow Q \text{ is the } \textbf{\textit{transition function}},}$[1]
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

A step in M is includes <u>both</u>:
- a step in M_1, and
- a step in M_2

Union is Closed For Regular Languages

<u>*Proof*</u>

- Given: $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, recognize A_1,
  $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, recognize A_2,
- Construct: $M = (Q, \Sigma, \delta, q_0, F)$, using M_1 and M_2, that recognizes $A_1 \cup A_2$
- states of M: $Q = \{(r_1, r_2) \mid r_1 \in Q_1 \text{ and } r_2 \in Q_2\}$ $= Q_1 \times Q_2$
  This set is the ***Cartesian product*** of sets Q_1 and Q_2
- M transition fn: $\delta\big((r_1, r_2), a\big) = \big(\delta_1(r_1, a), \delta_2(r_2, a)\big)$
- M start state: (q_1, q_2)

Start state of M is both start states of M_1 and M_2

Union is Closed For Regular Languages

Proof

- Given: $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$, recognize A_1,
  $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$, recognize A_2,
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- M start state: (q_1, q_2)

Remember:
Accept states must be subset of Q

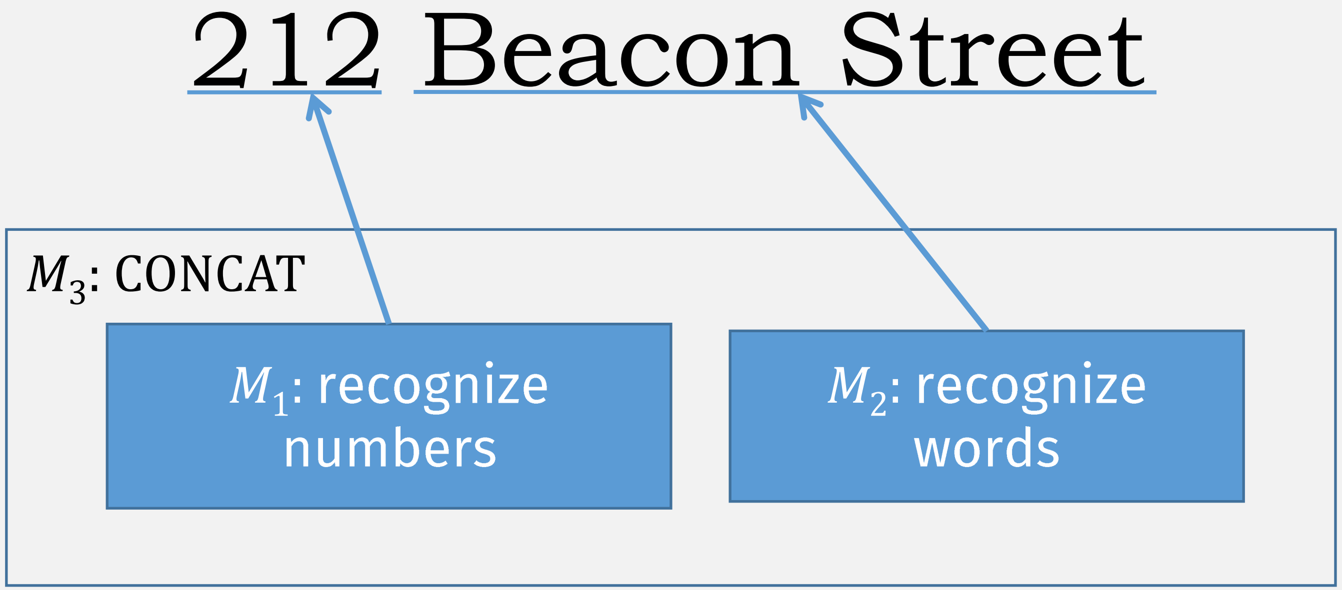
Accept if <u>either</u> M_1 or M_2 accept

- M accept states: $F = \{(r_1, r_2) | \, r_1 \in F_1 \boxed{\text{ or }} r_2 \in F_2\}$

(Q.E.D.) ■

Another operation: Concatenation

Example: **Recognizing street addresses**



Concatenation: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$

Concatenation of Languages

Let the alphabet Σ be the standard 26 letters $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$.

If $A = \{\mathtt{good}, \mathtt{bad}\}$ and $B = \{\mathtt{boy}, \mathtt{girl}\}$, then

$$A \circ B = \{\mathtt{goodboy}, \mathtt{goodgirl}, \mathtt{badboy}, \mathtt{badgirl}\}$$

Is Concatenation Closed?

THEOREM

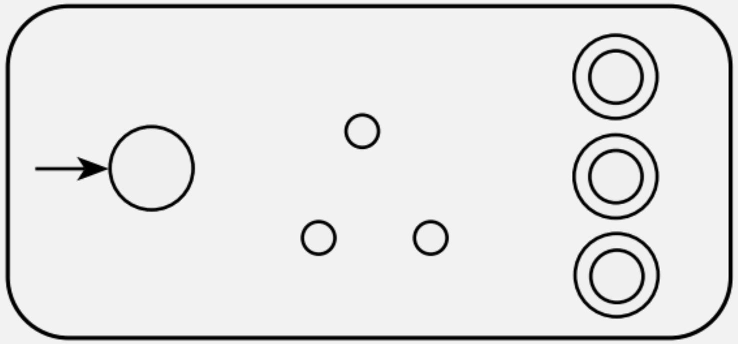
The class of regular languages is closed under the concatenation operation.

In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

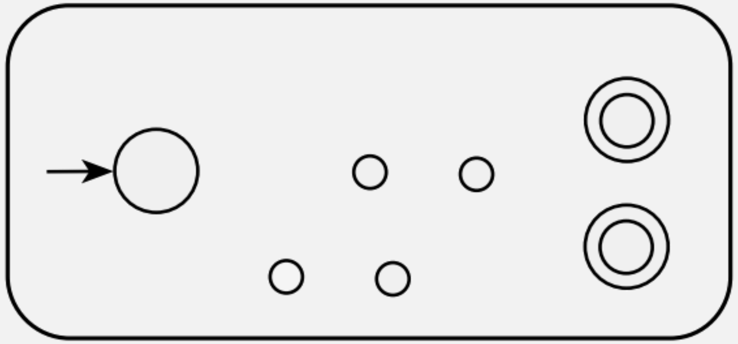
- **Construct a <u>new</u> machine M recognizing $A_1 \circ A_2$?** (like union)
  - Using **DFA M_1** (which recognizes A_1),
  - and **DFA M_2** (which recognizes A_2)

Concatentation

M_1



M_2

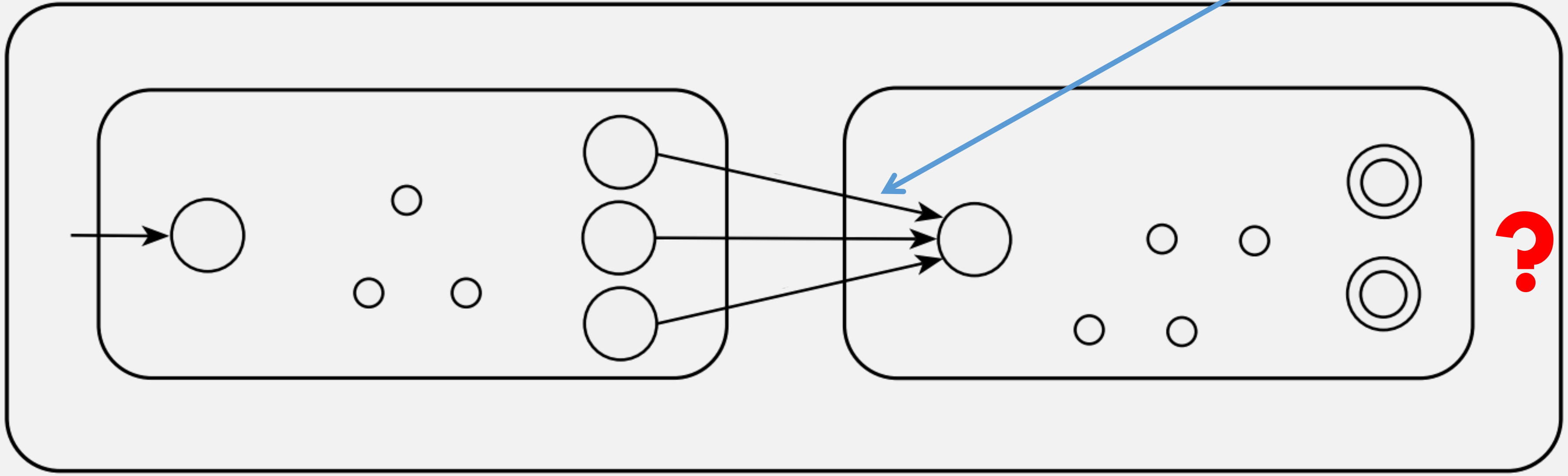


PROBLEM: Can only read input once, can't backtrack

Let M_1 recognize A_1, and M_2 recognize A_2.

Want: Construction of M to recognize $A_1 \circ A_2$

Need to switch machines at some point, but when?



???

Concatenation: $A \circ B = \{xy | \ x \in A \text{ and } y \in B\}$

Overlapping Concatenation Example

- Let M_1 recognize language A = { `jen`, `jens` }
- and M_2 recognize language B = { `smith` }
- Want: Construct M to recognize $A \circ B$ = { `jensmith`, `jenssmith` }

- If M sees `jen` ...
- M must decide to <u>either</u>:

Concatenation: $A \circ B = \{xy | \, x \in A \text{ and } y \in B\}$

Overlapping Concatenation Example

- Let M_1 recognize language A = { `jen`, $\boxed{\texttt{jens}}$ }
- and M_2 recognize language B = { $\boxed{\texttt{smith}}$ }
- Want: Construct M to recognize $A \circ B$ = { `jensmith`, `jenssmith` }

- If M sees `jen` …
- M must decide to <u>either</u>:
  - stay in M_1 (correct, if full input is $\boxed{\texttt{jens}}\boxed{\texttt{smith}}$)

Concatenation: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$

Overlapping Concatenation Example

- Let M_1 recognize language A = { `jen`, `jens` }
- and M_2 recognize language B = { `smith` }
- Want: Construct M to recognize $A \circ B$ = { `jensmith`, `jenssmith` }

A **DFA** can't do this!

- If M sees `jen` ...
- M must decide to <u>either</u>:
  - stay in M_1 (correct, if full input is `jenssmith`)
  - or switch to M_2 (correct, if full input is `jensmith`)
- <u>But</u> to recognize $A \circ B$, it needs to handle <u>both cases</u>!!
  - Without backtracking

Is Concatenation Closed?

FALSE?

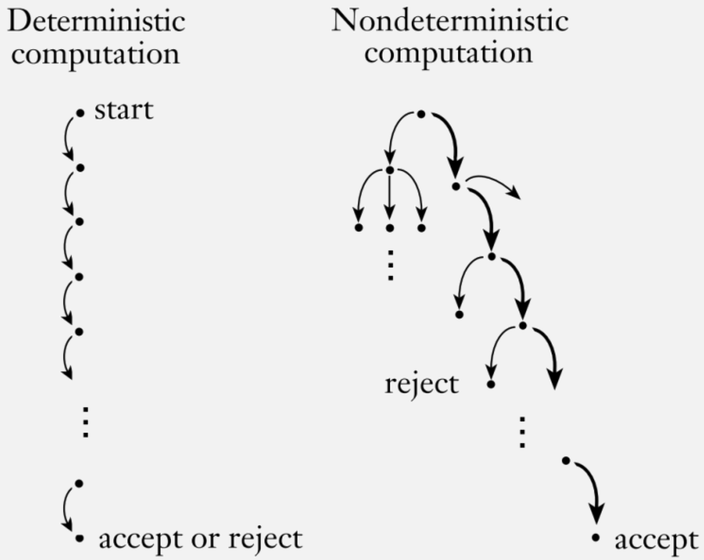
THEOREM

The class of regular languages is closed under the concatenation operation.

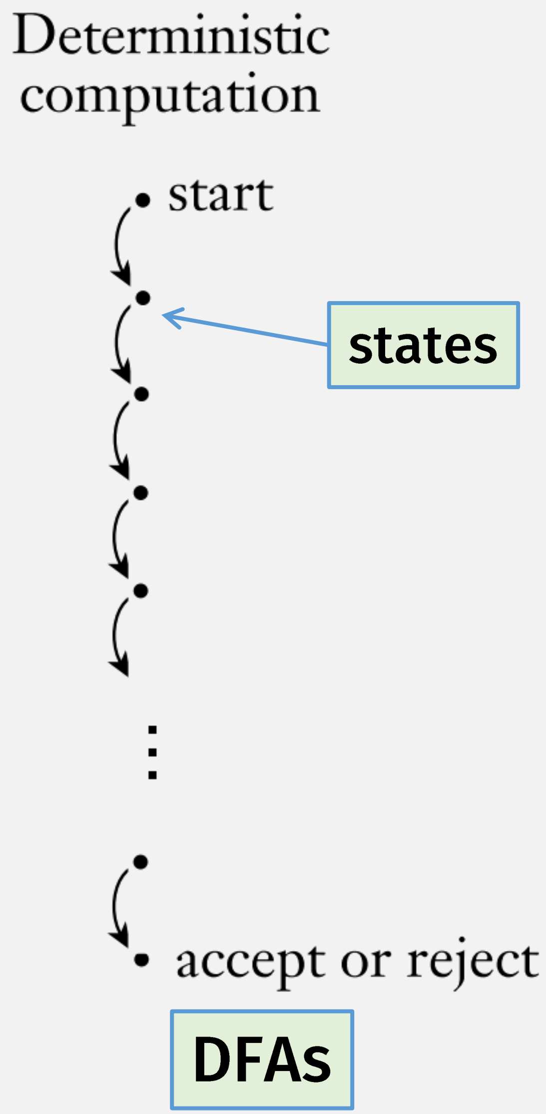
In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

- Cannot **combine A_1 and A_2's machine because:**
  - **Need to switch from A_1 to A_2 at some point …**
  - **… but we don't know when!** (we can only read input once)
- **This requires a new kind of machine!**
- But **does this mean concatenation is not closed for regular langs?**

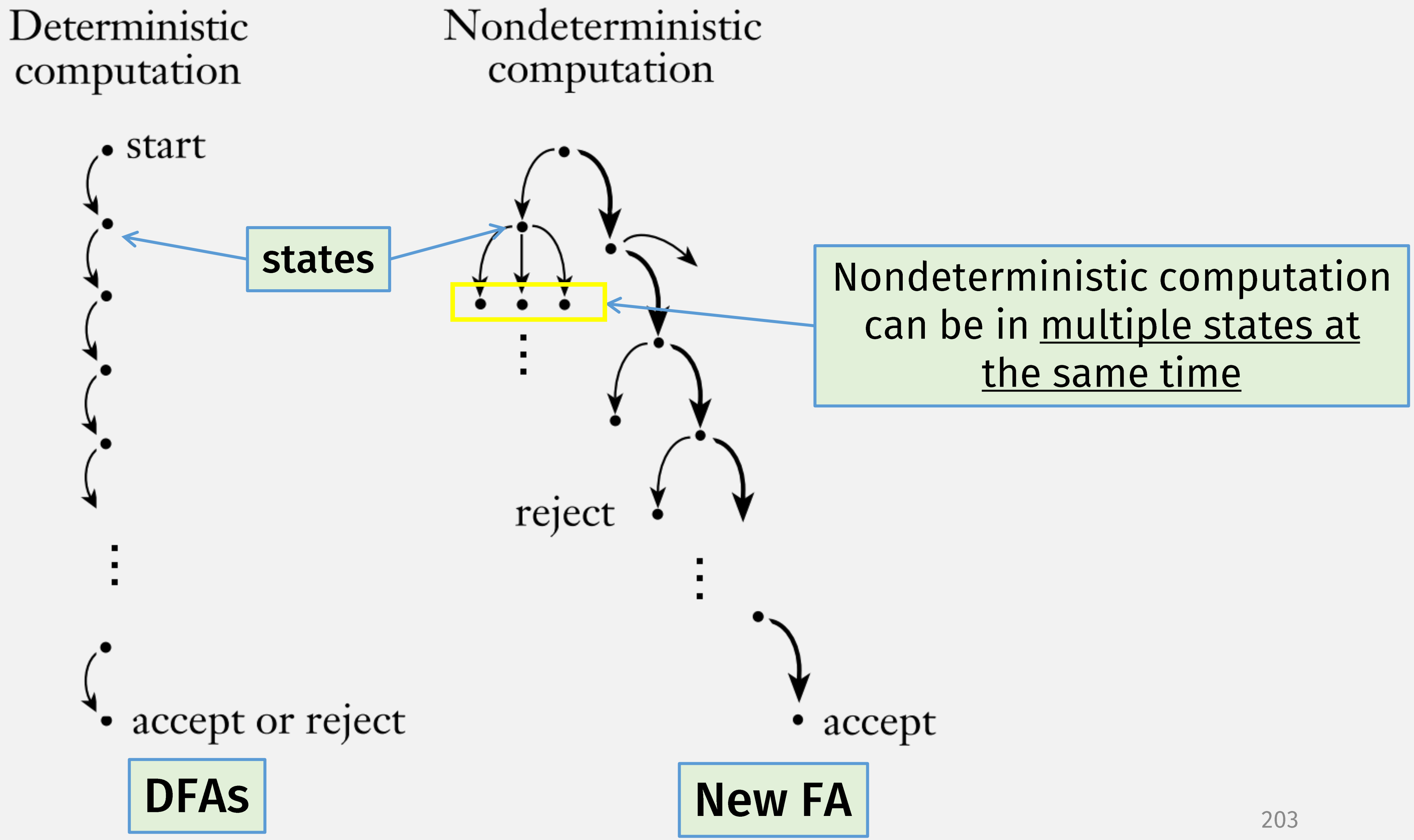
Nondeterminism



Deterministic vs Nondeterministic



Deterministic vs Nondeterministic



Finite Automata: The Formal Definition

DEFINITION

deterministic

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta : Q \times \Sigma \longrightarrow Q$ is the ***transition function***,
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

Also called a **Deterministic Finite Automata (DFA)**

Precise Terminology is Important

- A **finite automata** or **finite state machine** (**FSM**) defines …
  … computation with a finite number of states

- There are many kinds of FSMs

- We've learned one kind, the **Deterministic Finite Automata (DFA)**
  - (So currently, the terms DFA and FSM refer to the same definition)

- We will learn other kinds, e.g., **Nondeterministic Finite Automata (NFA)**

- Be careful with terminology!

Nondeterministic Finite Automata (NFA)

DEFINITION

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Compare with DFA:

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

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5. $F \subseteq Q$ is the ***set of accept states***.

Difference

Power set, i.e. a transition results in set of states

Power Sets

- A power set is the set of all subsets of a set

- Example: S = {**a**, **b**, **c**}

- Power set of S =
  - {{ }, {**a**}, {**b**}, {**c**}, {**a**, **b**}, {**a**, **c**}, {**b**, **c**}, {**a**, **b**, **c**}}
  - Note: includes the empty set!

Nondeterministic Finite Automata (NFA)

DEFINITION

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

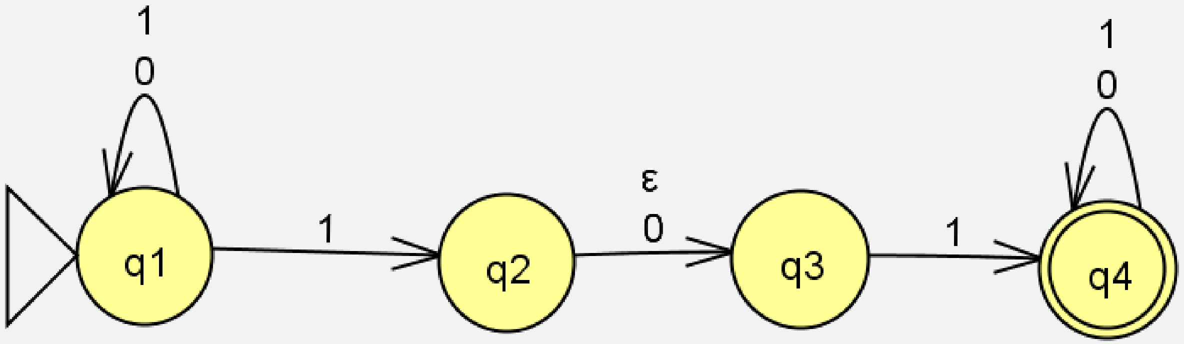
1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Transition label can be "empty", i.e., machine can transition without reading input

$$\boxed{\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}}$$

NFA Example

- Come up with a formal description of the following NFA:



DEFINITION

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2. Σ is a finite alphabet,
3. $\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where

$$\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$$

1. $Q = \{q_1, q_2, q_3, q_4\}$,
2. $\Sigma = \{0,1\}$,
3. δ is given as

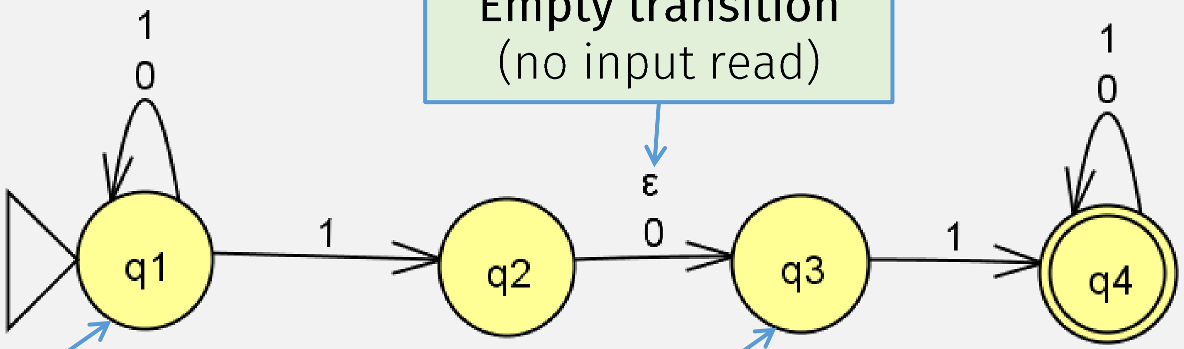
Empty transition (no input read)

	0	1	ε
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$,

Result of transition is a set

4. q_1 is the start state, and
5. $F = \{q_4\}$.

Empty transition (no input read)

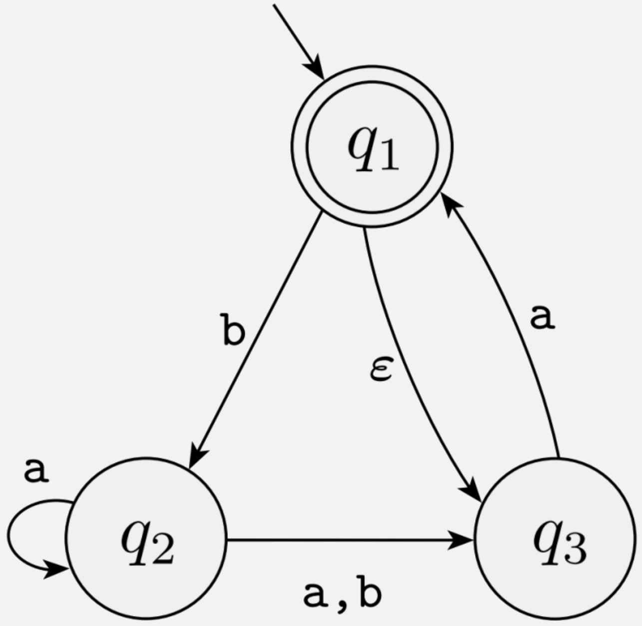


Multiple 1 transitions

No 0 transition

In-class Exercise

- Come up with a formal description for the following NFA
  - $\Sigma = \{ \texttt{a}, \texttt{b} \}$



DEFINITION

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta \colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

In-class Exercise Solution

Let $N = (Q, \Sigma, \delta, q_0, F)$

- $Q = \{ q_1, q_2, q_3 \}$
- $\Sigma = \{ \mathtt{a}, \mathtt{b} \}$
- δ ...
- $q_0 = q_1$
- $F = \{ q_1 \}$

$\delta(q_1, \mathtt{a}) = \{ \}$

$\delta(q_1, \mathtt{b}) = \{ q_2 \}$

$\delta(q_1, \varepsilon) = \{ q_3 \}$

$\delta(q_2, \mathtt{a}) = \{ q_2, q_3 \}$

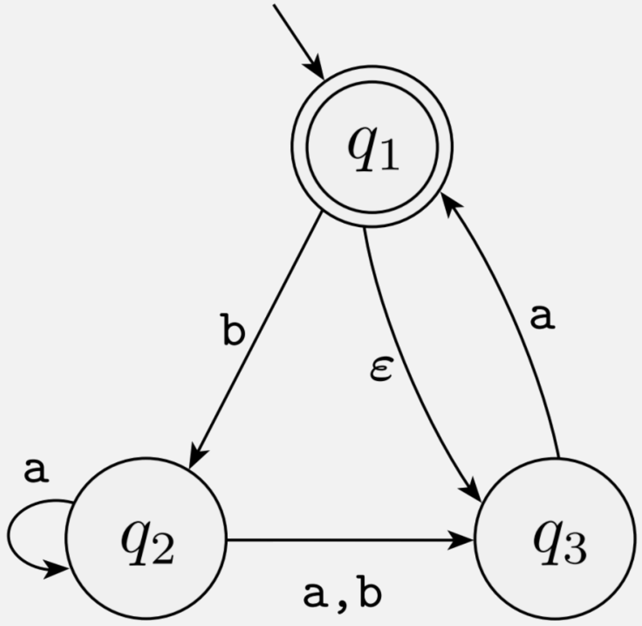
$\delta(q_2, \mathtt{b}) = \{ q_3 \}$

$\delta(q_2, \varepsilon) = \{ \}$

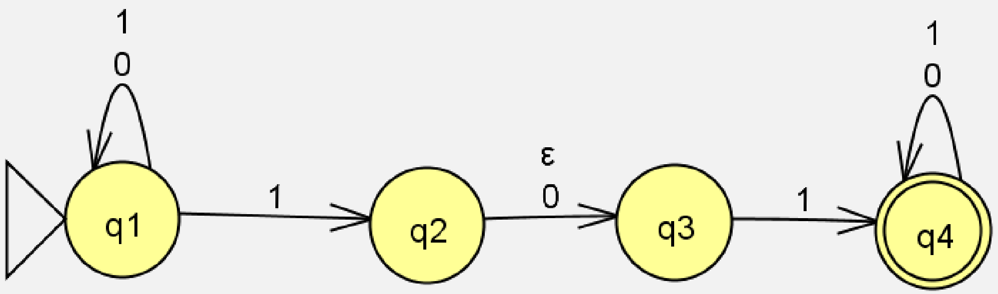
$\delta(q_3, \mathtt{a}) = \{ q_1 \}$

$\delta(q_3, \mathtt{b}) = \{ \}$

$\delta(q_3, \varepsilon) = \{ \}$



Next Time: Running Programs, NFAs (JFLAP demo): `010110`



Check-in Quiz 2/6

On gradescope