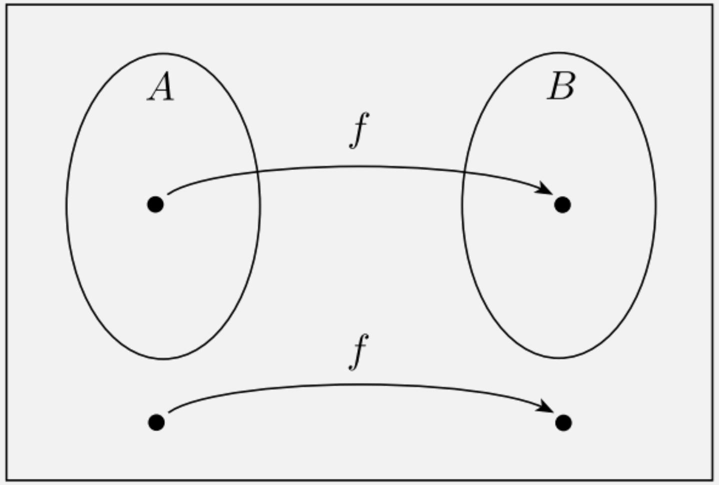
UMB CS 420

Mapping Reducibility

Wednesday, April 19, 2023



Announcements

- HW 9 still out
  - Due Sun 4/23 11:59pm EST

Quiz Preview

- Mapping reducibility is a relation between two …?

Last time: Undecidable …

No algorithm can compute **anything** about the language of a Turing Machine, i.e., about the runtime behavior of programs …

- $REGULAR_{\mathrm{TM}}$ = {<M> | M is a TM and $L(M)$ is a regular language}
- $CONTEXTFREE_{\mathrm{TM}}$ = {<M> | M is a TM and $L(M)$ is a CFL}
- $DECIDABLE_{\mathrm{TM}}$ = {<M> | M is a TM and $L(M)$ is a decidable language}
- $FINITE_{\mathrm{TM}}$ = {<M> | M is a TM and $L(M)$ is a finite language}
- …

Rice's Theorem

- $ANYTHING_{\mathrm{TM}}$ = {<M> | M is a TM and "… **anything** …" about $L(M)$}

Flashback: "Reduced"

$A_{\mathsf{TM}} = \{\langle M, w \rangle |\ M \text{ is a TM and } M \text{ accepts } w\}$ known

$\Downarrow$

$HALT_{\mathsf{TM}} = \{\langle M, w \rangle |\ M \text{ is a TM and } M \text{ halts on input } w\}$ unknown

Thm: $HALT_{\mathsf{TM}}$ is undecidable

Proof, by **contradiction**:

- Assume: $HALT_{\mathsf{TM}}$ **has** ***decider*** R**;** use it **to create** A_{TM} ***decider*****:**

Essentially, **we** convert **decidability of an** A_{TM} **string …**

$S =$ "On input $\langle M, w \rangle$, an encoding of a TM M and a string w:

1. Run TM R on input $\langle M, w \rangle$.
2. If R rejects, *reject*.
3. If R accepts, simulate M on w until it halts.
4. If M has accepted, *accept*; if M has rejected, *reject*."

Use R to first check if M will loop on w

Then run M on w knowing it won't loop

… into decidability of a $HALT_{\mathsf{TM}}$ **string**

A **potential** ***problem*****:** could the conversion itself go into an infinite loop?

- Contradicti[…] no decider!

Let's formalize this conversion, i.e., **mapping reducibilty**

Flashback: A_{NFA} is a decidable language

$A_{\mathsf{NFA}} = \{\langle B, w\rangle \mid B \text{ is an NFA that accepts input string } w\}$

Decider for A_{NFA} :

$N =$ "On input $\langle B, w\rangle$, where B is an NFA and w is a string:

1. Convert NFA B to an equivalent DFA C, using the procedure `NFA→DFA`
2. Run TM M on input $\langle C, w\rangle$.
3. If M accepts, *accept*; otherwise, *reject*."

We said this **NFA→DFA** algorithm is a decider TM, but **it doesn't accept/reject**?

More generally, **our analogy has been:**
"programs ~ TMs",
but **programs do more than accept/reject**?

Definition: Computable Functions

A function $f \colon \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

- A **computable function** is represented with a TM that, instead of accept/reject, "outputs" its final tape contents
- Example 1: All arithmetic operations
- Example 2: Converting between machines, like DFA→NFA
  - E.g., adding states, changing transitions, wrapping TM in TM, etc.

Definition: Mapping Reducibility

notation

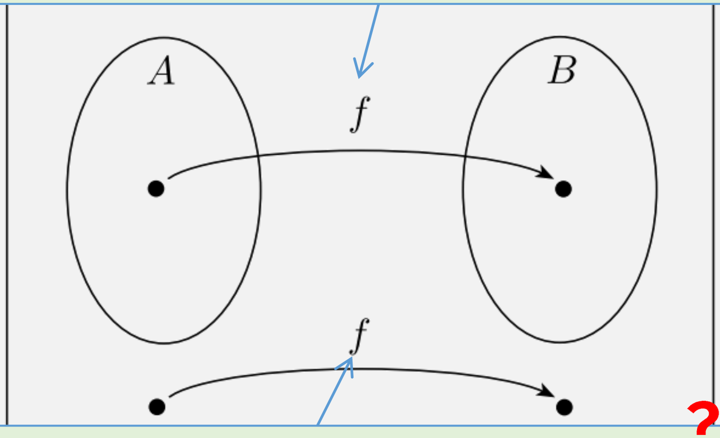
Language A is ***mapping reducible*** to language B, written $\boxed{A \leq_{\mathrm{m}} B}$, if there is a $\boxed{\text{computable function } f\colon \Sigma^* \longrightarrow \Sigma^*}$, where for every w,

$$\boxed{w \in A \iff f(w) \in B.}$$

$w \in A$
"if and only if"
$f(w) \in B$

The function f is called the ***reduction*** from A to B.

"forward" direction ($\Rightarrow$): if $w \in A$ then $f(w) \in B$



"reverse" direction ($\Leftarrow$): if $f(w) \in B$ then $w \in A$???

A function $f\colon \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Flashback: Equivalence of Contrapositive

"If X then Y" is equivalent to … ?

1. "If Y then X" (converse)

2. "If **not** X then **not** Y" (inverse)

3. "If **not** Y then **not** X" (contrapositive)

Flashback: Equivalence of Contrapositive

"If X then Y" is equivalent to … ?

× "If Y then X" (converse)

- **No!**

× "If **not** X then **not** Y" (inverse)

- **No!**

✓ "If **not** Y then **not** X" (contrapositive)

- **Yes!**

Definition: Mapping Reducibility

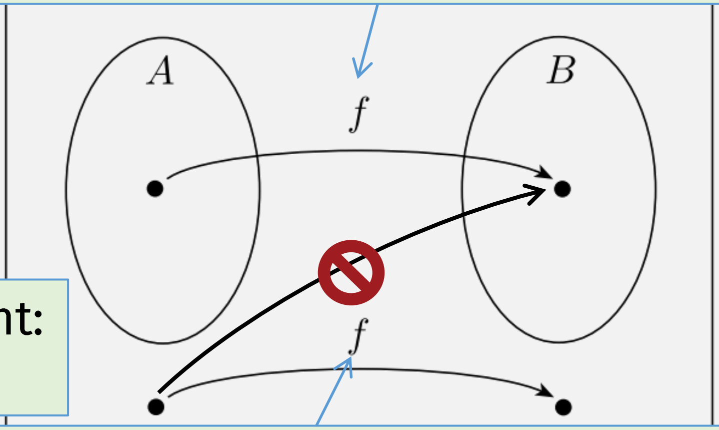
Language A is ***mapping reducible*** to language B, written $A \leq_{\mathrm{m}} B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$\boxed{w \in A \iff f(w) \in B.}$$

"if and only if"

The function f is called the ***reduction*** from A to B.

"forward" direction ($\Rightarrow$): if $w \in A$ then $f(w) \in B$



Reverse direction just as important: don't convert non-*A*s into *B*s

"reverse" direction ($\Leftarrow$): if $f(w) \in B$ then $w \in A$

Equivalent **(contrapositive)**: if $w \notin A$ then $f(w) \notin B$

Easier to prove

Proving Mapping Reducibility: 2 Steps

Step 1:
Show there is computable fn f... by creating a TM

Language A is ***mapping reducible*** to language B, written $A \leq_{\mathrm{m}} B$, if there is a $\boxed{\text{computable function } f\colon \Sigma^* \longrightarrow \Sigma^*}$, where for every w,

$$\boxed{w \in A \iff f(w) \in B.}$$

"if and only if"

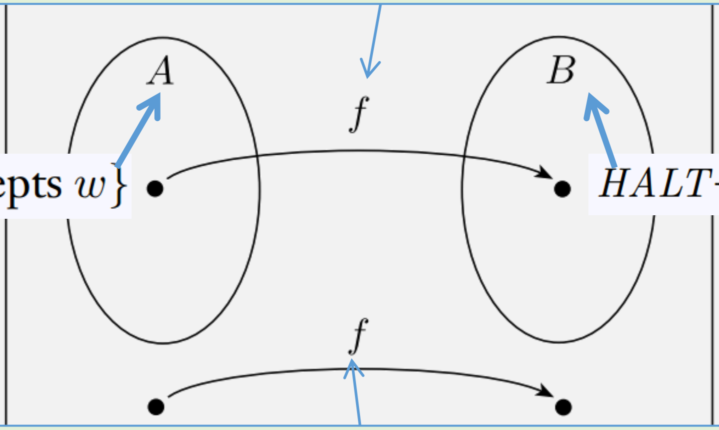
The function f is called the ***reduction*** from A to B.

Step 2:
Prove the iff is true for that computable fn TM

Step 2a: "forward" direction (⇒): if $w \in A$ then $f(w) \in B$

e.g.

$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$



$HALT_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$

Step 2b: "reverse" direction (⇐): if $f(w) \in B$ then $w \in A$

Step 2b, alternate (contrapositive): if $w \notin A$ then $f(w) \notin B$

A function $f\colon \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Thm: A_{TM} is mapping reducible to $HALT_{\mathsf{TM}}$

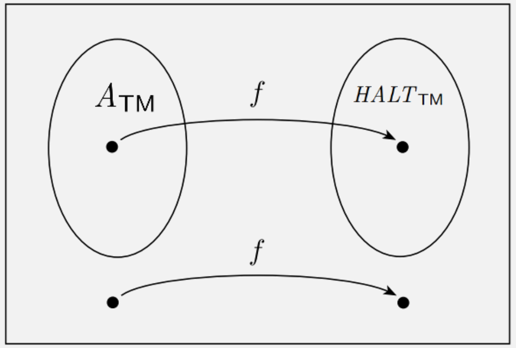
$A_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$

$HALT_{\mathsf{TM}} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$

To show: $A_{\mathsf{TM}} \leq_{\mathrm{m}} HALT_{\mathsf{TM}}$

Step 1: **create computable fn** f: $<M, w> \rightarrow <M', w>$ where:

Step 2: **show** $\langle M, w\rangle \in A_{\mathsf{TM}}$ if and only if $\langle M', w\rangle \in HALT_{\mathsf{TM}}$



The following machine F computes a reduction f.

$F =$ "On input $\langle M, w\rangle$:

1. Construct the following machine M'.
   $M' =$ "On input x:
   1. Run M on x.
   2. If M accepts, *accept*.
   3. If M rejects, enter a loop."
2. Output $\langle M', w\rangle$."

Converts M to M'

Step 2:
M accepts *w*
if and only if
M' halts on *w*

Output new *M'*

M' is like *M*, except it always loops when it doesn't accept

Language A is ***mapping reducible*** to language B, written $A \leq_{\mathrm{m}} B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \iff f(w) \in B.$$

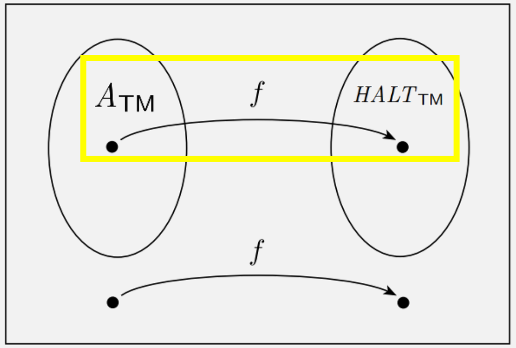
The function f is called the ***reduction*** from A to B.

A function $f\colon \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

☑ ⇒ If M accepts w, then M' halts on w

- M' accepts (and thus halts) if M accepts

⇐ If M' halts on w, then M accepts w



The following machine F computes a reduction f.

$F =$ "On input $\langle \boxed{M}, w\rangle$:

1. Construct the following machine M'.

   $\boxed{M'} =$ "On input x:

   1. Run M on x.
   2. If M accepts, *accept*.
   3. If M rejects, enter a loop."

2. Output $\langle M', w\rangle$."

If M accepts this string

Then M' accepts it (and halts)

Step 2:
M accepts w
if and only if
M' halts on w

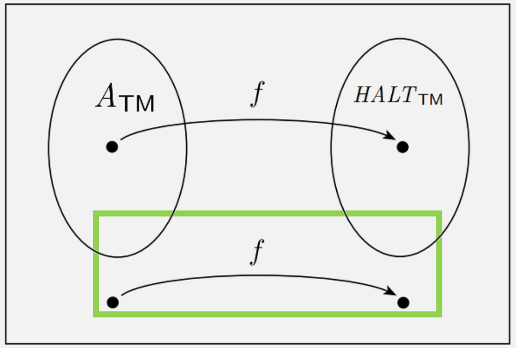
☑ ⇒ If M **accepts** w, then M' **halts on** w

- M' **accepts** (and thus **halts**) **if** M **accepts**

⇐ If M' **halts on** w, then M **accepts** w

☑ ⇐ (Alternatively) If M **doesn't accept** w, then M' **doesn't halt on** w (contrapositive)

- **Two possibilities for "doesn't accept":**
  1. M loops: M' loops and doesn't halt
  2. M rejects: M' loops and doesn't halt



The following machine F computes a reduction f.

F = "On input $\langle \boxed{M}, w\rangle$:
  1. Construct the following machine M'.
  $\boxed{M'}$ = "On input x:
    1. Run M on x.
    2. If M accepts, *accept*.
    3. If M rejects, enter a loop."
  2. Output $\langle M', w\rangle$."

If M **loops**, then M' **loops**

If M **rejects**, then M' **loops**

<u>Step 2</u>:
M **accepts** w
if and only if
M' **halts on** w

Now we know what **mapping reducibility** is, and **how to prove it for two languages**; but **what is it** <u>used for</u>?

Uses of Mapping Reducibility

- To prove **Decidability**
- To prove **Undecidability**

Thm: If $A \leq_m B$ and B is decidable, then A is decidable.

Has a decider

Must create decider

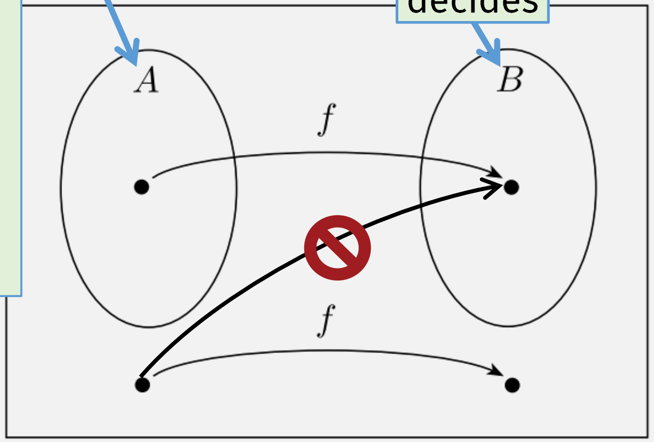
PROOF We let M be the decider for B and f be the reduction from A to B. We describe a decider N for A as follows.

N = "On input w:
1. Compute $f(w)$.
2. Run M on input $f(w)$ and output whatever M outputs."

decides

decides

We know this is true bc of the iff (specifically the reverse direction)



Why is it true that:
If M accepts $f(w)$ then N should accept w ??
i.e., $f(w)$ in B guarantees that w in A???

Language A is ***mapping reducible*** to language B, written $A \leq_m B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \iff f(w) \in B.$$

The function f is called the ***reduction*** from A to B.

Corollary: If $A \leq_m B$ and A is undecidable, then B is undecidable.

- Proof by **contradiction**.
- Assume B **is decidable.**
- **Then A is decidable** (by the previous thm).
- Contradiction: **we already said A is undecidable**

If $A \leq_m B$ and B is decidable, then A is decidable.

Summary: Showing Mapping Reducibility

Step 1:
Show there is computable fn f ... by creating a TM

Language A is ***mapping reducible*** to language B, written $A \leq_{\mathrm{m}} B$, if there is a $\boxed{\text{computable function } f\colon \Sigma^* \longrightarrow \Sigma^*}$, where for every w,

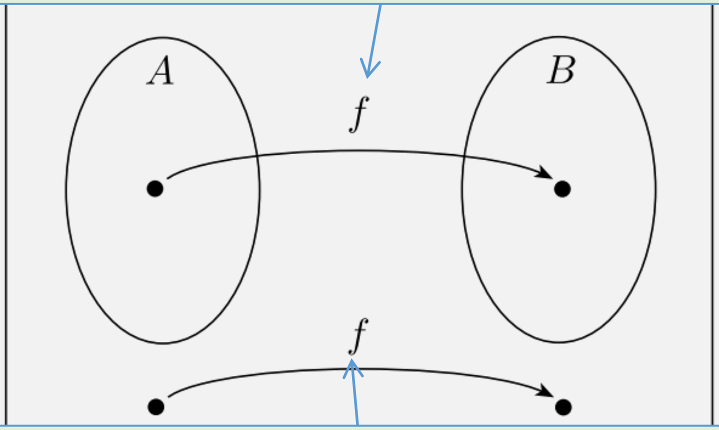
$$\boxed{w \in A \iff f(w) \in B.}$$

"if and only if"

Step 2:
Prove the iff is true

The function f is called the ***reduction*** from A to B.

Step 2a: "forward" direction (⇒): if $w \in A$ then $f(w) \in B$



Step 2b: "reverse" direction (⇐): if $f(w) \in B$ then $w \in A$

Step 2b, alternate (contrapositive): if $w \notin A$ then $f(w) \notin B$

A function $f\colon \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Summary: Using Mapping Reducibility

To prove decidability …

- If $A \leq_{\mathrm{m}} B$ and B is decidable, then A is decidable.

Known

Unknown
(want to prove)

To prove undecidability …

- If $A \leq_{\mathrm{m}} B$ and A is undecidable, then B is undecidable.

Undecidability Proof Technique #4:
Mapping Reducibility
+ this theorem

Be careful with the direction of the **reduction,**
i.e., what is known and what is unknown!

Alternate Proof: The Halting Problem

$HALT_{\mathsf{TM}}$ is undecidable

- If $A \leq_{\mathrm{m}} B$ and A is undecidable, then B is undecidable.

Must be known

- $A_{\mathsf{TM}} \leq_{\mathrm{m}} HALT_{\mathsf{TM}}$

Undecidability Proof Technique #4: **Mapping Reducibility** + this theorem

- Since A_{TM} is undecidable,
- … and we showed mapping reducibility from A_{TM} to $HALT_{\mathsf{TM}}$,
- then $HALT_{\mathsf{TM}}$ is undecidable ■

Flashback: EQ_{TM} is undecidable

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Proof by **contradiction**:

- Assume EQ_{TM} **has *decider* R;** use it **to create** E_{TM} ***decider*:**

$= \{\langle M\rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

S = "On input $\langle M\rangle$, where M is a TM:

1. Run R on input $\langle M, M_1\rangle$, where M_1 is a TM that rejects all inputs.
2. If R accepts, *accept*; if R rejects, *reject*."

Alternate Proof: EQ_{TM} is undecidable

$EQ_{\mathsf{TM}} = \{\langle M_1, M_2\rangle |\ M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Show **mapping reducibility:** $E_{\mathsf{TM}} \leq_{\mathrm{m}} EQ_{\mathsf{TM}}$

Step 1: **create computable fn** f: $<M> \rightarrow <M_1, M_2>$, **computed by** S

$S =$ "On input $\langle M\rangle$, where M is a TM:

1. Construct: $\langle M, M_1\rangle$, where M_1 is a TM that rejects all inputs.

2. Output: $\langle M, M_1\rangle$

Step 2: **show iff requirements of mapping reducibility** (hw exercise)

And use theorem …

Undecidability Proof Technique #4:
Mapping Reducibility + theorem ☟

If $A \leq_{\mathrm{m}} B$ and A is undecidable, then B is undecidable.

Flashback: E_{TM} is undecidable

$E_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

Proof, by **contradiction**:

- Assume E_{TM} **has** ***decider*** *R*; use it **to create** A_{TM} ***decider***:

> $S =$ "On input $\langle M, w\rangle$, an encoding of a TM M and a string w:
> **1.** Use the description of M and w to construct the TM M_1
>
> > $M_1 =$ "On input x:
> > **1.** If $x \neq w$, *reject*.
> > **2.** If $x = w$, run M on input w and *accept* if M does."
>
> **2.** Run R on input $\langle M_1 \rangle$.
> **3.** If R accepts, *reject*; if R rejects, *accept*."

- So **this only reduces** A_{TM} **to** $\overline{E_{\mathsf{TM}}}$

If M accepts w,
then M_1 accepts w,
meaning M_1 is not in E_{TM}!

Alternate Proof: E_{TM} is undecidable

$E_{\mathsf{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

Show **mapping reducibility??:** $A_{\mathsf{TM}} \leq_{\mathrm{m}} E_{\mathsf{TM}}$

Step 1: create computable fn *f*: <*M*, *w*> → <*M'*>, computed by *S*

$S =$ "On input $\langle M, w \rangle$, an encoding of a TM M and a string w:

1. Use the description of M and w to construct the TM M_1

   $M_1 =$ "On input x:
   1. If $x \neq w$, *reject*.
   2. If $x = w$, run M on input w and *accept* if M does."

2. Output: $\langle M_1 \rangle$.
3. ~~If R accepts, *reject*; if R rejects, *accept*.~~"

If *M* accepts *w*, then M_1 accepts *w*, meaning M_1 is not in E_{TM}!

- So this only reduces A_{TM} to $\overline{E_{\mathsf{TM}}}$
- It's good enough! Still proves E_{TM} is undecidable
  - If … undecidable langs are closed under **complement**

Step 2: show iff requirements of mapping reducibility (hw exercise)

Language Complement

Complement (OPPO from hw3) of a language A, written $\overline{A}$ ...

... is the set of all strings not in set A

Example:

$E_{\mathsf{TM}} = \{ \langle M \rangle \mid M \text{ is a } \mathsf{TM} \text{ and } L(M) = \emptyset \}$

$\overline{E_{\mathsf{TM}}} = \{ \langle M \rangle \mid M \text{ is a } \mathsf{TM} \text{ and } L(M) \neq \emptyset \}$

$\cup \{ w \mid w \text{ is a string that is not a } \mathsf{TM} \text{ description} \}$

Undecidable Langs Closed under Complement

Proof by **contradiction**

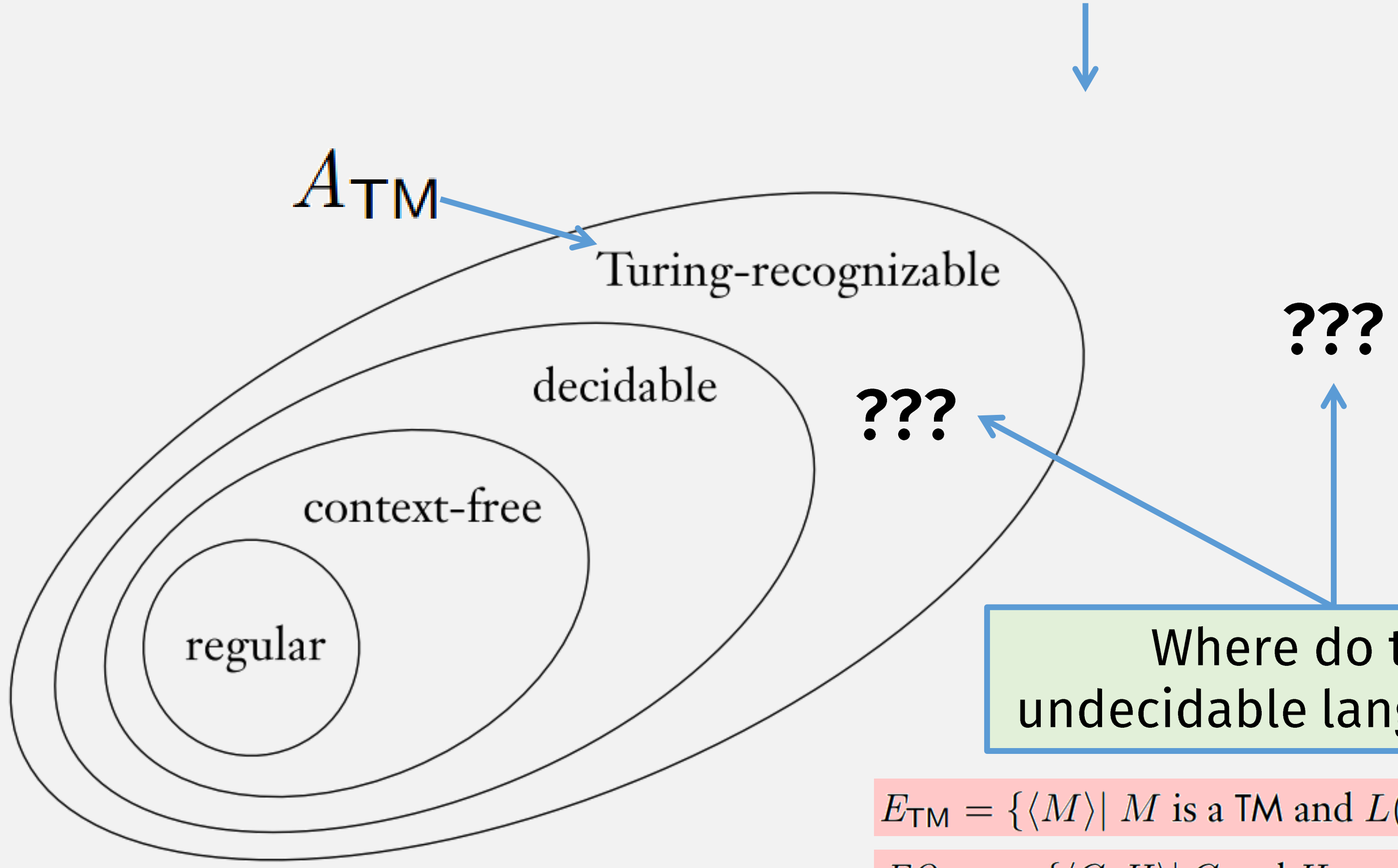
- Assume some lang L is undecidable and $\overline{L}$ is decidable …
  - Then $\overline{L}$ has a decider

Contradiction!

- … then we can create decider for L from decider for $\overline{L}$ …
  - Because **decidable languages are closed under complement** (hw10?)!

Next Time: Turing Unrecognizable?

Is there anything out here?



Where do these undecidable languages go?

$E_{\text{TM}} = \{\langle M \rangle \mid M \text{ is a TM and } L(M) = \emptyset\}$

$EQ_{\text{CFG}} = \{\langle G, H \rangle \mid G \text{ and } H \text{ are CFGs and } L(G) = L(H)\}$

$EQ_{\text{TM}} = \{\langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are TMs and } L(M_1) = L(M_2)\}$

Check-in Quiz 4/19

On gradescope