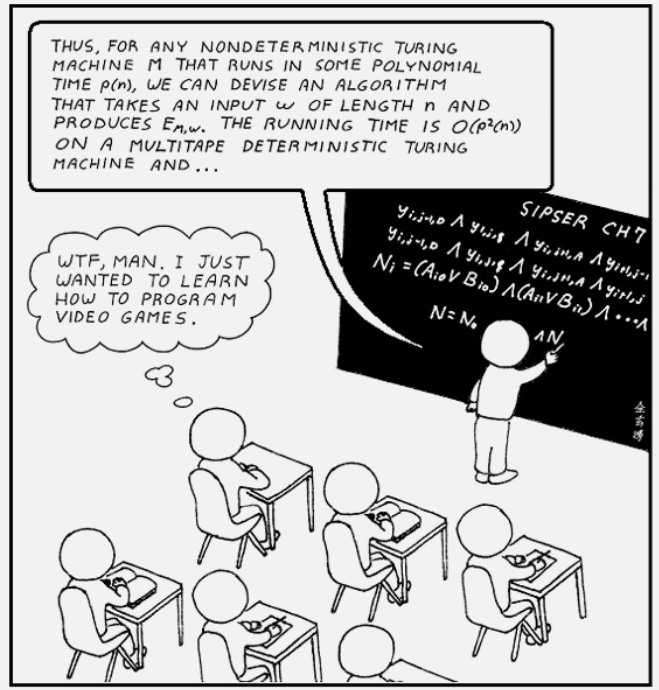
UMB CS420

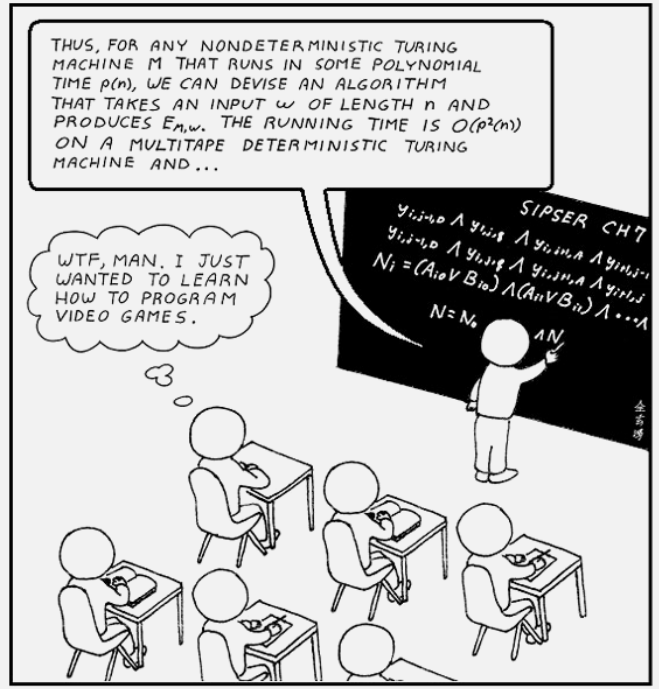
Nondeterministic TMs

Wednesday, April 3, 2024



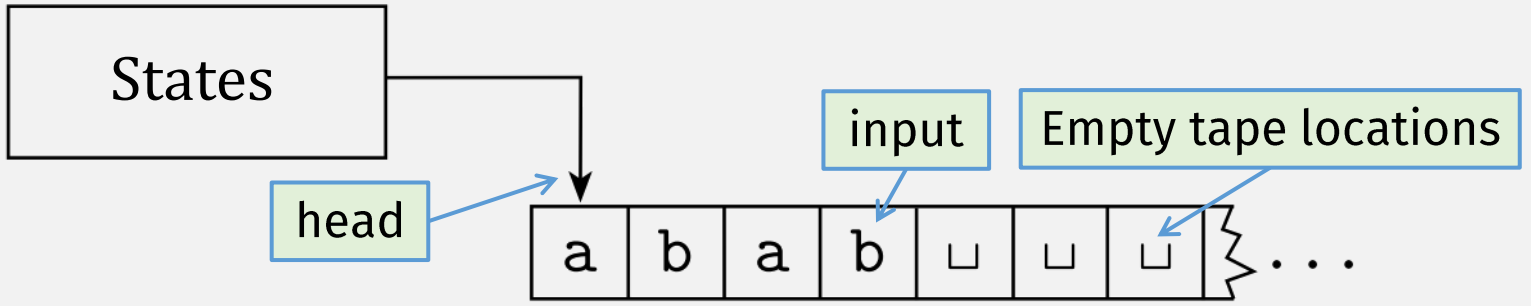
Announcements

- HW 7 out
  - due Mon 4/8 12pm noon EST



Last Time: Turing Machines

- **Turing Machines** can read and write to arbitrary "tape" cells
  - Tape initially contains input string
- The tape is infinite
  - (to the right)



- On a transition, "head" can move left or right 1 step

> Call a language ***Turing-recognizable*** if some Turing machine recognizes it.

Turing Machine: High-Level Description

- M_1 accepts if input is in language $B = \{w\#w \mid w \in \{0,1\}^*\}$

M_1 = "On input string w:

1. Zig-zag across th ding positions on either side of the # s ne the same symb n Cross off symbo d to keep track of which symbols correspond.
2. When all symbols to the check for any remaining s symbols remain, *reject*; ot

We will (mostly) define TMs using **high-level descriptions,** like this one

(But it must always correspond to some formal **low-level tuple** description)

Analogy:
High-level (e.g., Python) function definitions
vs
Low-level assembly language

Turing Machines: Formal Tuple Definition

A ***Turing machine*** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where Q, Σ, Γ are all finite sets and

1. Q is the set of states,
2. Σ is the input alphabet not containing the ***blank symbol*** $\sqcup$,
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. $\delta : Q \times \Gamma \longrightarrow Q \times \Gamma \times \{\text{L}, \text{R}\}$ is the transition function,
5. $q_0 \in Q$ is the start state,
6. $q_{\text{accept}} \in Q$ is the accept state, and
7. $q_{\text{reject}} \in Q$ is the reject state, where $q_{\text{reject}} \neq q_{\text{accept}}$.

read write move

Flashback: DFAs vs NFAs

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta: Q \times \Sigma \longrightarrow Q$ is the ***transition function***,
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

vs

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta: Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Nondeterministic transition produces set of possible next states

Remember: Turing Machine Formal Definition

A ***Turing machine*** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where Q, Σ, Γ are all finite sets and

1. Q is the set of states,
2. Σ is the input alphabet not containing the ***blank symbol*** $\sqcup$,
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. $\delta : Q \times \Gamma \longrightarrow Q \times \Gamma \times \{\text{L}, \text{R}\}$ is the transition function,
5. $q_0 \in Q$ is the start state,
6. $q_{\text{accept}} \in Q$ is the accept state, and
7. $q_{\text{reject}} \in Q$ is the reject state, where $q_{\text{reject}} \neq q_{\text{accept}}$.

Nondeterministic

Remember:

Turing Machine Formal Definition

Nondeterministic Turing Machine

A is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where Q, Σ, Γ are all finite sets and

1. Q is the set of states,
2. Σ is the input alphabet not containing the ***blank symbol*** $\sqcup$,
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. ~~$\delta: Q \times \Gamma \longrightarrow Q \times \Gamma \times \{\text{L}, \text{R}\}$~~ $\delta: Q \times \Gamma \longrightarrow \mathcal{P}(Q \times \Gamma \times \{\text{L}, \text{R}\})$
5. $q_0 \in Q$ is the start state,
6. $q_{\text{accept}} \in Q$ is the accept state, and
7. $q_{\text{reject}} \in Q$ is the reject state, where $q_{\text{reject}} \neq q_{\text{accept}}$.

Thm: Deterministic TM ⇔ Non-det. TM

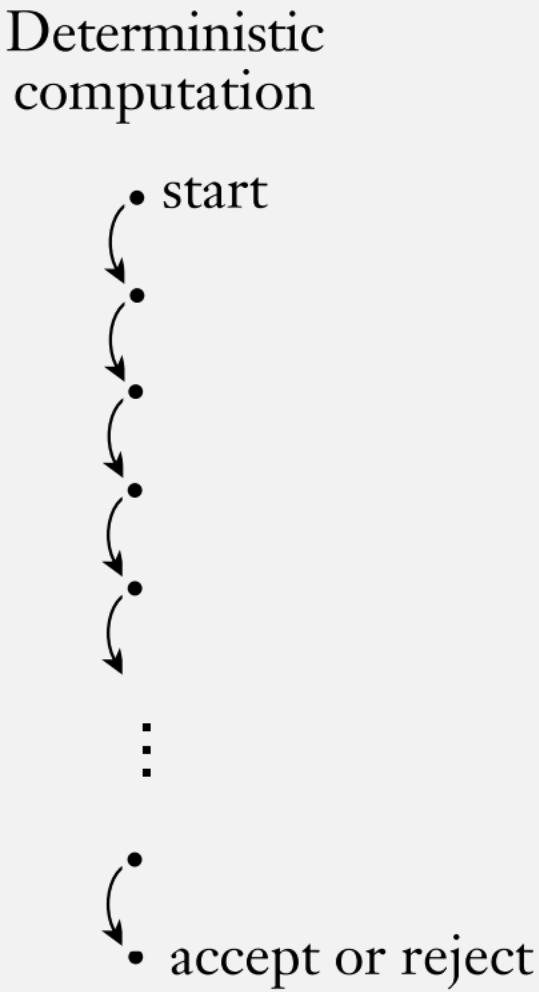
⇒ If a **deterministic TM** recognizes a language, then a **non-deterministic TM** recognizes the language

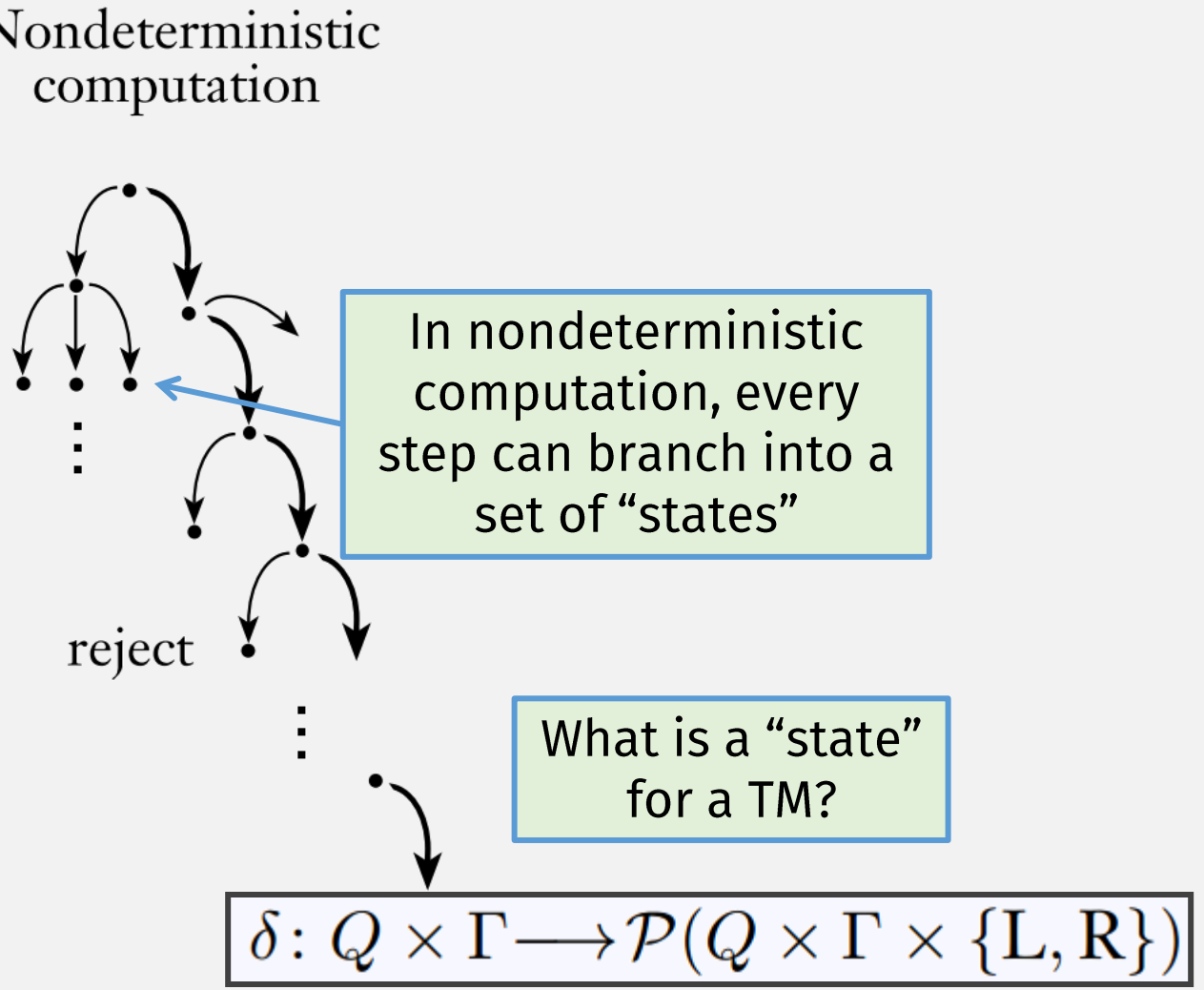
- Convert: Deterministic TM → Non-deterministic TM …
- … change Deterministic TM δ fn output to a one-element set
  - $\delta_{\text{ntm}}(q, a) = \{\delta_{\text{dtm}}(q, a)\}$
  - (just like conversion of DFA to NFA --- HW 3, Problem 1)
- **DONE!**

⇐ If a **non-deterministic TM** recognizes a language, then a **deterministic TM** recognizes the language

- Convert: Non-deterministic TM → Deterministic TM …
- … ???

Review: Nondeterminism





Flashback: PDA Configurations (IDs)

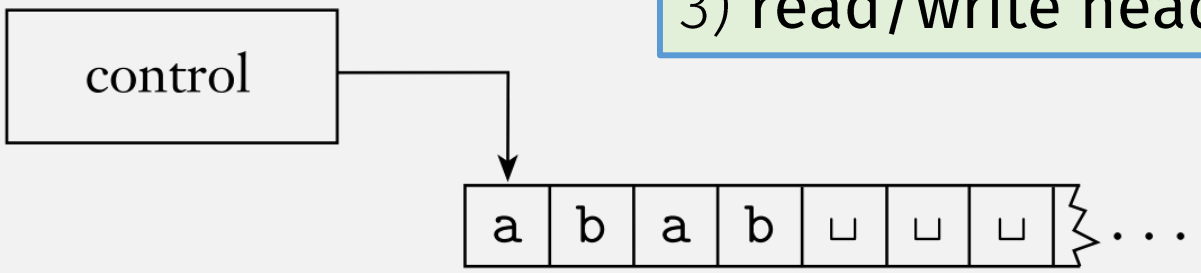
- A **configuration** (or **ID**) is a "snapshot" of a PDA's computation

- 3 components (q, w, γ) :
  q = the **current state**
  w = the **remaining input string**
  γ = the **stack contents**

A **sequence of configurations** represents a **PDA** computation

TM Configuration (ID) = ???

1) states



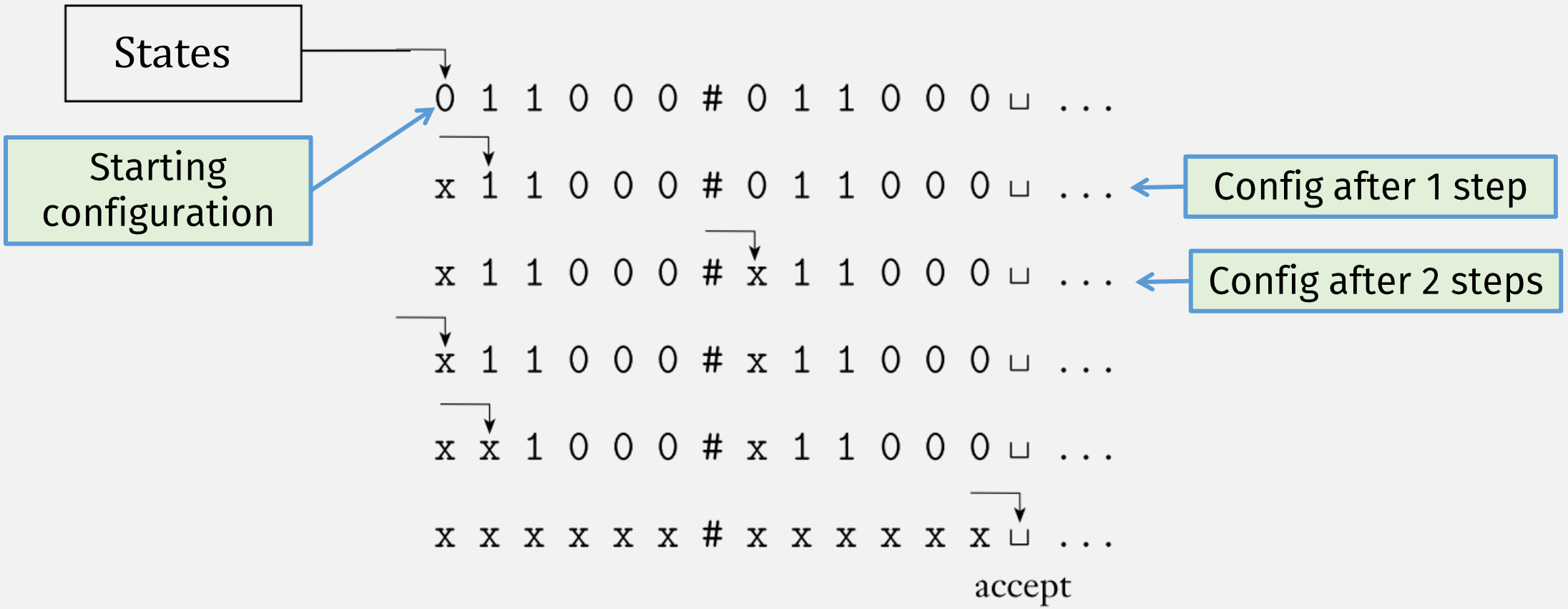
3) read/write head

2) Tape contents

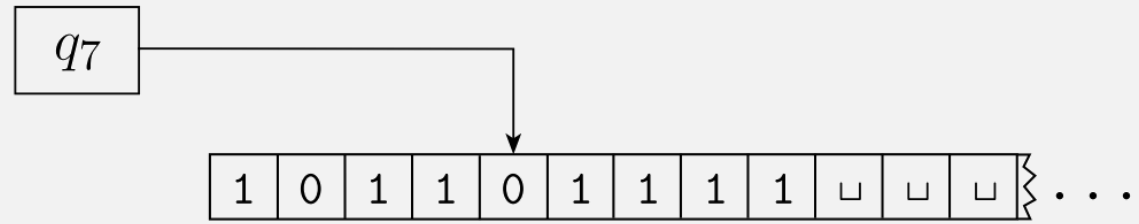
A ***Turing machine*** is a 7-tuple, $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{accept}}, q_{\text{reject}})$, where Q, Σ, Γ are all finite sets and

1. Q is the set of states,
2. Σ is the input alphabet not containing the ***blank symbol*** $\sqcup$,
3. Γ is the tape alphabet, where $\sqcup \in \Gamma$ and $\Sigma \subseteq \Gamma$,
4. $\delta\colon Q \times \Gamma \longrightarrow Q \times \Gamma \times \{\mathrm{L}, \mathrm{R}\}$ is the transition function,
5. $q_0 \in Q$ is the start state,
6. $q_{\text{accept}} \in Q$ is the accept state, and
7. $q_{\text{reject}} \in Q$ is the reject state, where $q_{\text{reject}} \neq q_{\text{accept}}$.

TM Configuration = State + Head + Tape



TM Configuration = State + Head + Tape



$$1011q_701111$$

Textual representation of "configuration" (use this in HW)

1st char after state is current head position

TM Computation, Formally

$$M = (Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$$

Single-step

(Right) $\alpha q_1 \mathbf{a} \beta \vdash \alpha \mathbf{x} q_2 \beta$

head

Next config (head moved past written char)

write

read

if $q_1, q_2 \in Q$

$\delta(q_1, \mathbf{a}) = (q_2, \mathbf{x}, \mathrm{R})$

$\mathbf{a}, \mathbf{x} \in \Gamma \quad \alpha, \beta \in \Gamma^*$

(Left) $\alpha b q_1 \mathbf{a} \beta \vdash \alpha q_2 b \mathbf{x} \beta$

head

if $\delta(q_1, \mathbf{a}) = (q_2, \mathbf{x}, \mathrm{L})$

(wrote **x** and) head moved left

Multi-step

- Base Case

$I \vdash^* I$ for any ID I

- Recursive Case

$I \vdash^* J$ if there exists some ID K such that $I \vdash K$ and $K \vdash^* J$

Edge cases:

$q_1 \mathbf{a} \beta \vdash q_2 \mathbf{x} \beta$ if $\delta(q_1, \mathbf{a}) = (q_2, \mathbf{x}, \mathrm{L})$

Head stays at leftmost cell

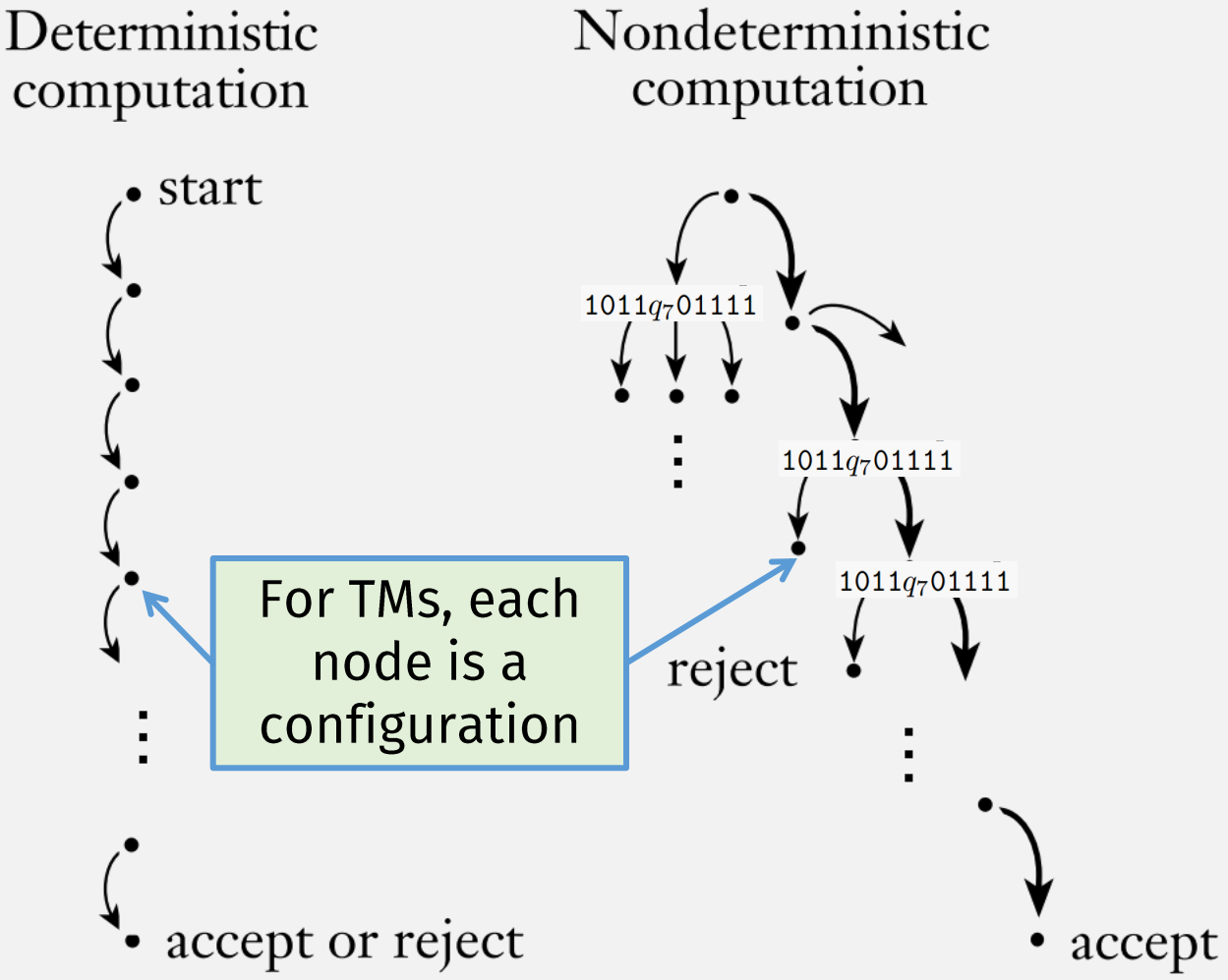
(L move, when already at leftmost cell)

$\alpha q_1 \vdash \alpha \sqcup q_2$ if $\delta(q_1, \sqcup) = (q_2, \sqcup, \mathrm{R})$

Add blank symbol to config

(R move, when at rightmost filled cell)

Nondeterminism in TMs

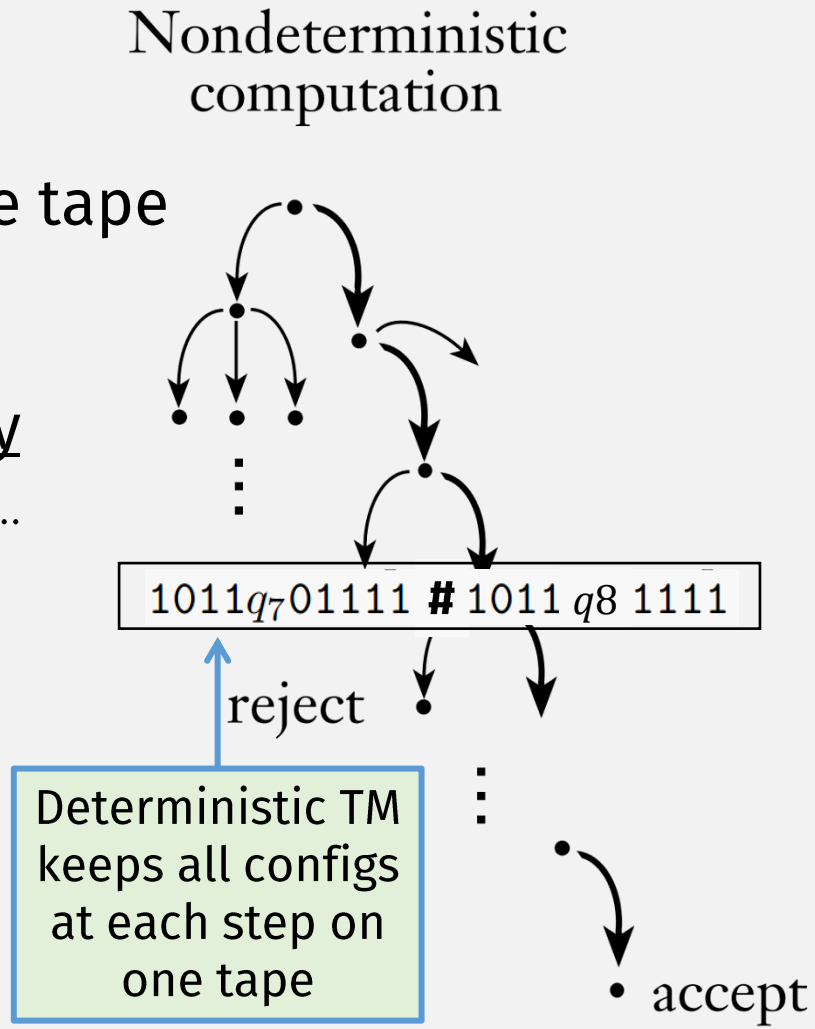


Nondeterministic TM → Deterministic

1st way

- Simulate NTM with Det. TM:
  - Det. TM keeps multiple configs on single tape
    - Like how single-tape TM simulates multi-tape
  - Then run all computations, concurrently
    - I.e., 1 step on one config, 1 step on the next, …
  - Accept if any accepting config is found
  - **Important:**
    - Why must we step configs concurrently?

Because any one path can go on forever!



Interlude: Running TMs inside other TMs

Remember analogy: **TMs** are **like function definitions, they** can be "*called*" **like functions …**

Exercise:

- Given: TMs M_1 and M_2
- Create: TM $\boldsymbol{M}$ that **accepts** if either M_1 or M_2 accept

Possible solution #1:

$\boldsymbol{M}$ = on input x,

1. `Call` M_1 with arg x; **accept** x if M_1 accepts
2. `Call` M_2 with arg x; **accept** x if M_2 accepts

Possible Results for M

M_1	M_2	$\boldsymbol{M}$
reject	accept	accept
accept	reject	accept
accept	loops	accept
loops	accept	loops

Note: This solution would be ok if we knew M_1 and M_2 were **deciders** (which halt on all inputs)

"loop" means input string not accepted (but it should be)

Interlude: Running TMs inside other TMs

Just an analogy: "*calling*" TMs actually means "computing" its computation …

Exercise:

- Given: TMs M_1 and M_2
- Create: TM $\boldsymbol{M}$ that **accepts** if either M_1 or M_2 accept

… with concurrency!

Possible solution #1:

M = on input x,

1. Call M_1 with arg x; accept x if M_1 accepts
2. Call M_2 with arg x; accept x if M_2 accepts

M_1	M_2	M
reject	accept	accept
accept	reject	accept
accept	loops	accept
loops	accept	loops

Possible solution #2:

$\boldsymbol{M}$ = on input x,

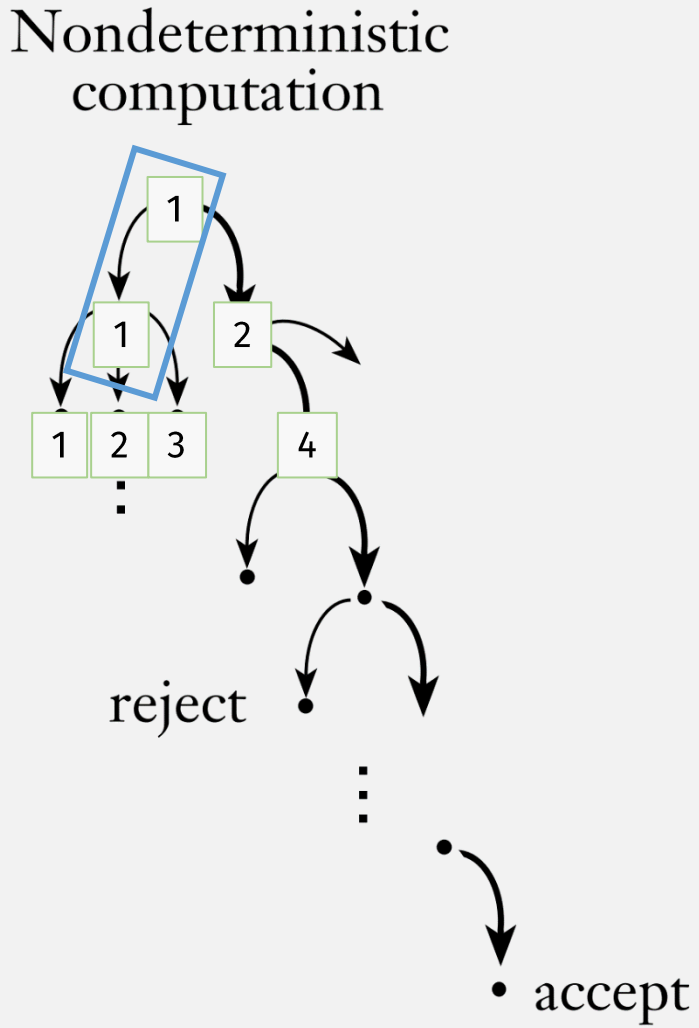
1. Call M_1 and M_2, each with x, concurrently, i.e.,
   a) Run M_1 with x for 1 step; **accept** x if M_1 accepts
   b) Run M_2 with x for 1 step; **accept** x if M_2 accepts
   c) Repeat

M_1	M_2	$\boldsymbol{M}$
reject	accept	**accept**
accept	reject	**accept**
accept	loops	**accept**
loops	accept	**accept**

Nondeterministic TM → Deterministic

2nd way (Sipser)

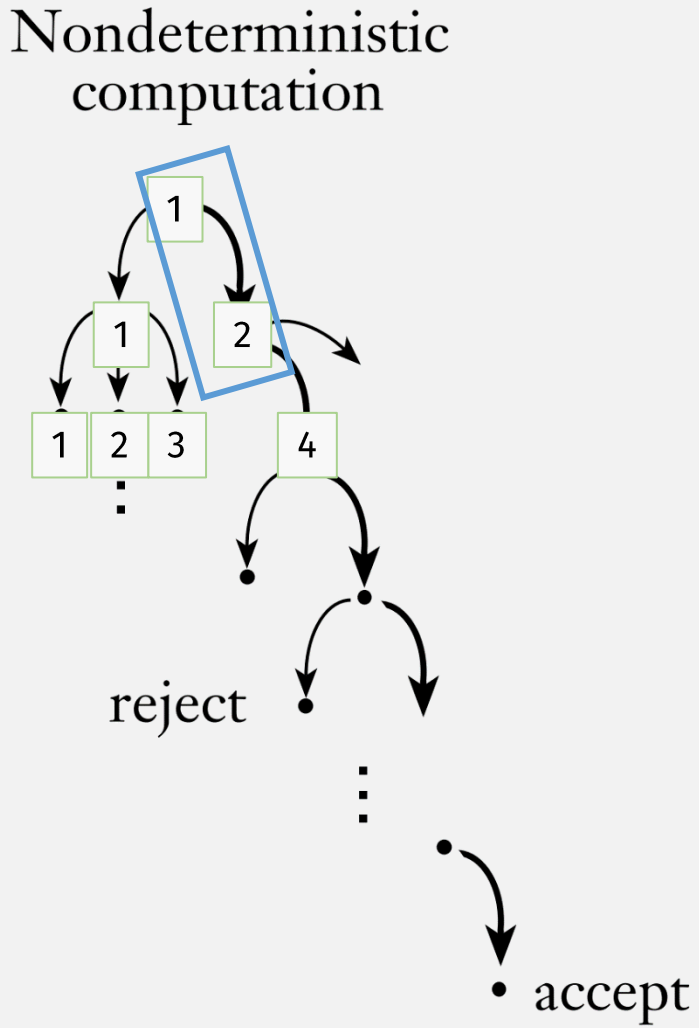
- Simulate NTM with Det. TM:
  - Number the nodes at each step
  - Check all tree paths (in breadth-first order)
    - 1
    - **1-1**



Nondeterministic TM → Deterministic

2nd way (Sipser)

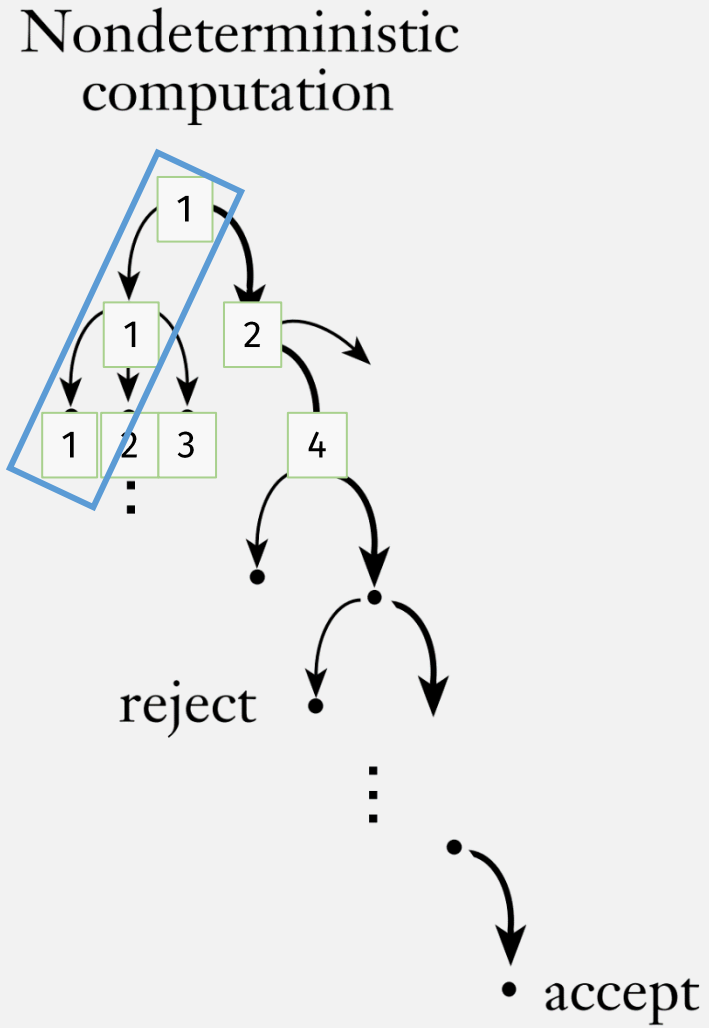
- Simulate NTM with Det. TM:
  - Number the nodes at each step
  - Check all tree paths (in breadth-first order)
    - 1
    - 1-1
    - 1-2



Nondeterministic TM → Deterministic

2nd way (Sipser)

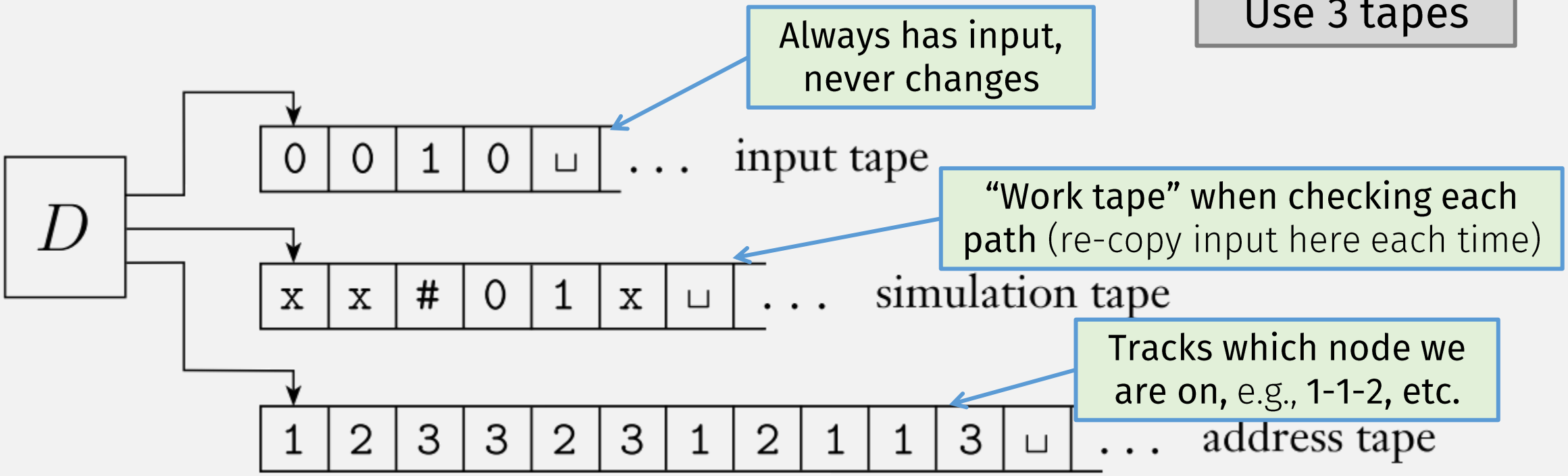
- **Simulate NTM with Det. TM:**
  - Number the nodes at each step
  - Check all tree paths (in breadth-first order)
    - 1
    - 1-1
    - 1-2
    - **1-1-1**



Nondeterministic TM → Deterministic

2nd way (Sipser)

Use 3 tapes



Nondeterministic TM ⇔ Deterministic TM

☑ ⇒ If a deterministic TM recognizes a language,
then a nondeterministic TM recognizes the language

- Convert Deterministic TM → Non-deterministic TM

☑ ⇐ If a nondeterministic TM recognizes a language,
then a deterministic TM recognizes the language

- Convert Nondeterministic TM → Deterministic TM

■

Conclusion: These are All Equivalent TMs!

- Single-tape Turing Machine
- Multi-tape Turing Machine
- Non-deterministic Turing Machine

Interlude: Running TMs inside other TMs

Just an analogy: "***calling***" TMs actually means "computing" its computation …

Exercise:

- Given: TMs M_1 and M_2
- Create: TM $\boldsymbol{M}$ that **accepts** if either M_1 or M_2 accept

Possible solution #1:

M = on input x,

1. `Call` M_1 with arg x; accept x if M_1 accepts
2. `Call` M_2 with arg x; accept x if M_2 accepts

M_1	M_2	M
reject	accept	accept
accept	reject	accept
accept	loops	accept
loops	accept	loops

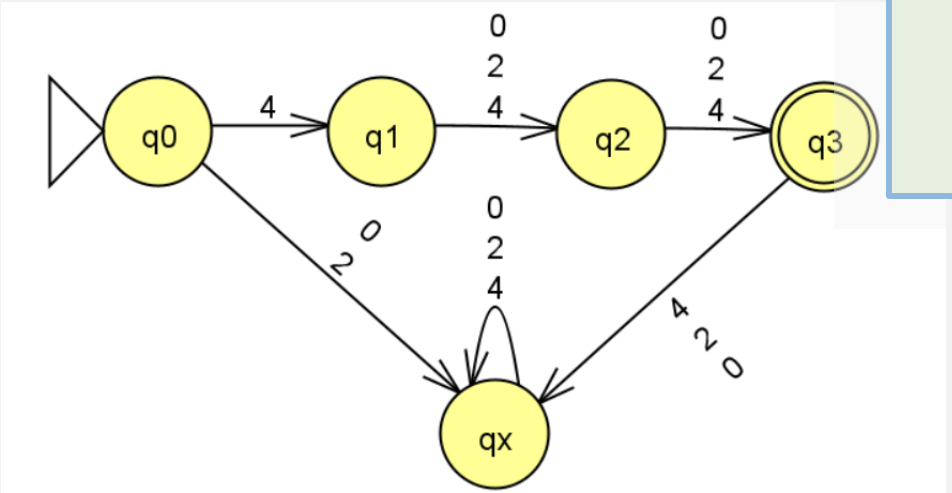
Possible solution #2:

$\boldsymbol{M}$ = on input x,

1. `Call` M_1 and M_2, each with x, concurrently, i.e.,
   a) Run M_1 with x for 1 step; **accept** x if M_1 accepts
   b) Run M_2 with x for 1 step; **accept** x if M_2 accepts
   c) Repeat

M_1	M_2	$\boldsymbol{M}$
reject	accept	**accept**
accept	reject	**accept**
accept	loops	**accept**
loops	accept	**accept**

Flashback: HW 1, Problem 1



1. Come up with 2 strings that are accepted by the DFA. These strings are said to be in the **language** rec

2. Come up with
not in the lan

3. Come up with a formal description for this DFA.

   Recall that a DFA's formal description is a tuple of five components, e.g. $M = (Q, \Sigma, \delta, q_{start}, F)$.

   You may assume that the alphabet contains only the symbols from the diagram.

4. Then for each of the following, say whether the computation represents an **accepting computation** or not (make sure to review the definition of an accepting computation). If the answer is no, explain why not:

   a. $\hat{\delta}(q0, 420)$

Figuring out this HW problem about a DFA's computation … is itself (meta) computation!

~~language~~

What kind of computation is it?

Could you write a program (function) to do it?

Remember:
TMs = program (functions)

A function: `DFAaccepts(B,w)` returns TRUE if DFA `B` accepts string `w`

1) `Define` "current" state q_{current} = start state q_0
2) `For` each input char a_i … in w
   a) `Define` $q_{\text{next}} = \delta_B(q_{\text{current}}, a_i)$
   b) `Set` $q_{\text{current}} = q_{\text{next}}$
3) `Return` TRUE if q_{current} is an accept state (of `B`)

You had to "compute" how a DFA computes

This is "computing" the **accepting computation** $\hat{\delta}(q_0, w) \in F$!!

The language of **DFAaccepts**

The set of strings that a **Turing Machine** accepts is a **language** …

$$A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$$

Is this language a set of strings???

A function: **DFAaccepts(B,w)** returns TRUE if DFA **B** accepts string **w**

Interlude: Encoding Things into Strings

Definition: A Turing machine's input is always a **string**

Problem: We sometimes want TM's (program's) input to be "something else" …

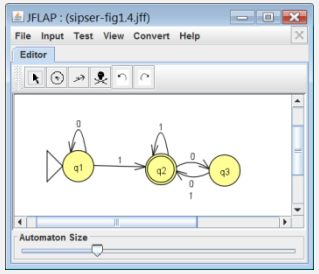
- **set, graph, DFA, …?**

Solution: allow **encoding** other kinds of TM input as a string

Notation: <SOMETHING> = string **encoding** for SOMETHING

- A tuple combines multiple encodings, e.g., $\langle B, w \rangle$ (from prev slide)

Example: Possible string encoding for a DFA?



```
<automaton>
  <!--The list of states.-->
  <state name="q1"><initial/></state>
  <state name="q2"><final/></state>
  <state name="q3"></state>
  <!--The list of transitions.-->
  <transition>
    <from>0</from>
    <to>0</to>
    <read>0</read>
  </transition>
  <transition>
    <from>1</from>
```

Details don't matter! (In this class) Just assume it's possible

Or:

$$(Q, \Sigma, \delta, q_0, F)$$

(written as string)

Interlude: High-Level TMs and Encodings

A high-level TM description:

1. Needs to say the **type** of its input
   - E.g., graph, DFA, etc.

M = "On input $\langle B, w \rangle$, where B is a DFA and w is a string:

2. Doesn't need to say <u>how</u> input string is encoded

3. Assumes TM <u>knows</u> how to parse and extract parts of input

Description of M can refer to B's $(Q, \Sigma, \delta, q_0, F)$

4. Assumes input is a <u>valid</u> encoding
   - Invalid encodings implicitly rejected

`DFAaccepts` as a TM recognizing A_{DFA}

Remember:
TM ~ program (function)
Creating TM ~ programming

$$A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$$

A function: `DFAaccepts(B,w)`
returns TRUE if DFA **B** accepts string **w**

1) `Define` "current" state q_{current} = start state q_0
2) `For` each input char a_i ... in w
   a) `Define` $q_{\text{next}} = \delta(q_{\text{current}}, a_i)$
   b) `Set` $q_{\text{current}} = q_{\text{next}}$
3) `Return` TRUE if q_{current} is an accept state

TM M_{DFA} =

"On input $\langle B, w\rangle$, where B is a DFA and w is a string:

$B = (Q, \Sigma, \delta, q_0, F)$

1) `Define` "current" state q_{current} = start state q_0
2) `For` each input char a_i ... in w
   a) `Define` $q_{\text{next}} = \delta(q_{\text{current}}, a_i)$
   b) `Set` $q_{\text{current}} = q_{\text{next}}$
3) **Accept** if q_{current} is an accept state

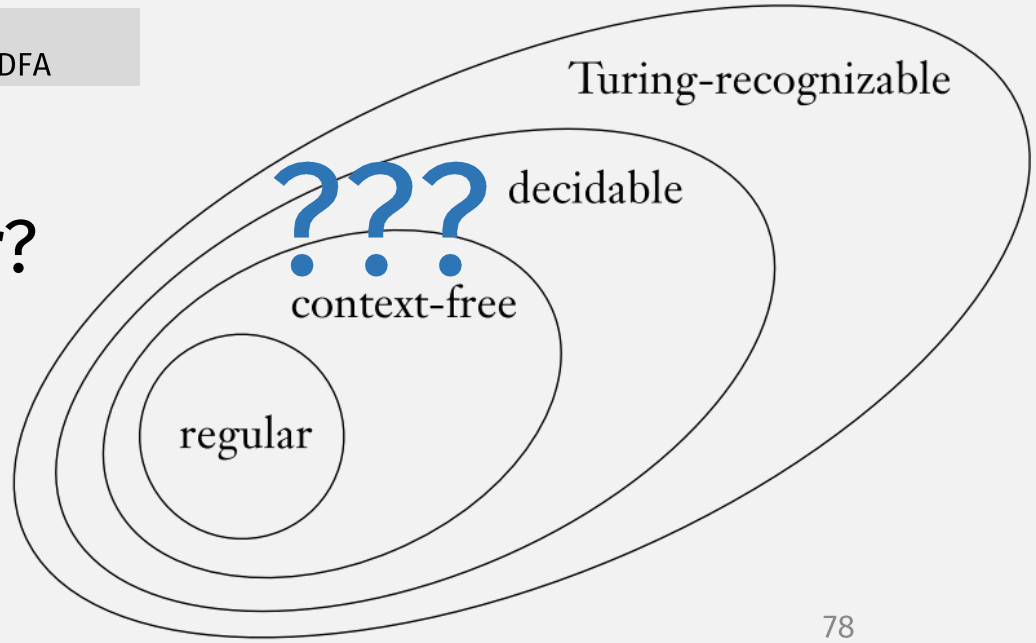
The language of **DFAaccepts**

$$A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$$

TM M_{DFA}

- A_{DFA} has a Turing machine
- But is that TM a **decider** or **recognizer**?
  - I.e., is it an **algorithm**?
- To show it's an algo, need to <u>prove</u>:

A_{DFA} is a decidable language



How to prove that a language is decidable?

How to prove that a language is decidable?

Statements	**Justifications**
1. If a **decider** decides a lang L, then L is a **decidable** lang	1. Definition of **decidable** langs
2. Define **decider** M = On input w … , M **decides** L (Key step)	2. See M def, and Examples Table
3. L is a **decidable** language	3. By statements #1 and #2

How to Design Deciders

- A **Decider** is a TM …
  - See previous slides on how to:
    - write a **high-level TM description**
    - Express **encoded** input strings
  - E.g., M = On input <B, w>, where B is a DFA and w is a string: …

- A **Decider** is a TM … that must always **halt**
  - Can only **accept** or **reject**
  - Cannot go into an infinite loop

- So a **Decider** definition must include an extra **termination argument**:
  - Explains how every step in the TM halts
  - (Pay special attention to loops)

- Remember our analogy: TMs ~ Programs … so *Creating* a TM ~ Programm*ing*
  - To design a TM, think of how to write a program (function) that does what you want

Next Time: A_{DFA} is a decidable language

$$A_{\mathsf{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$$

Decider for A_{DFA} :