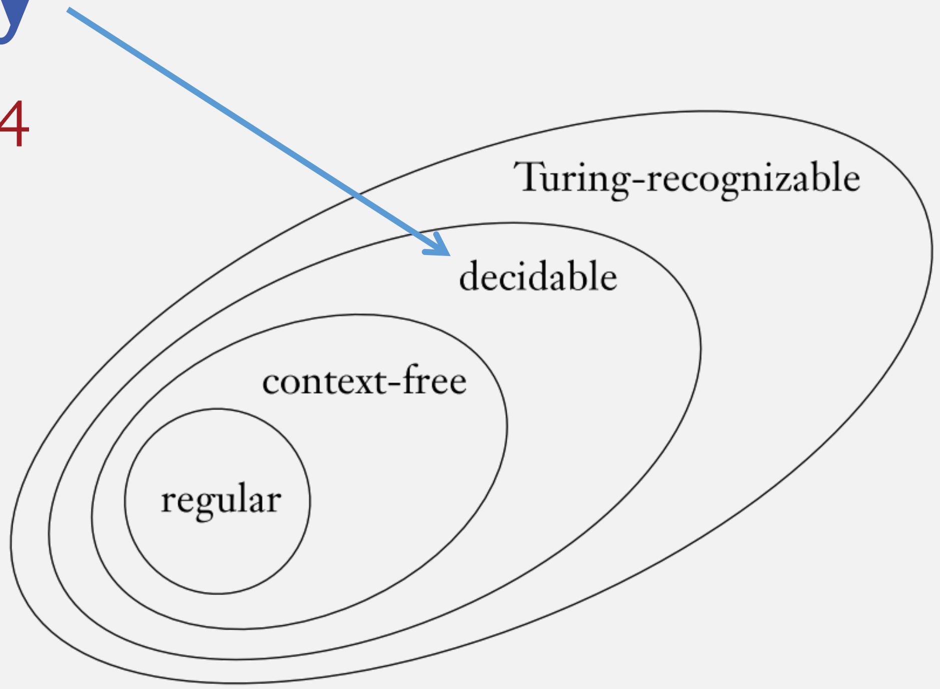
UMB CS 420

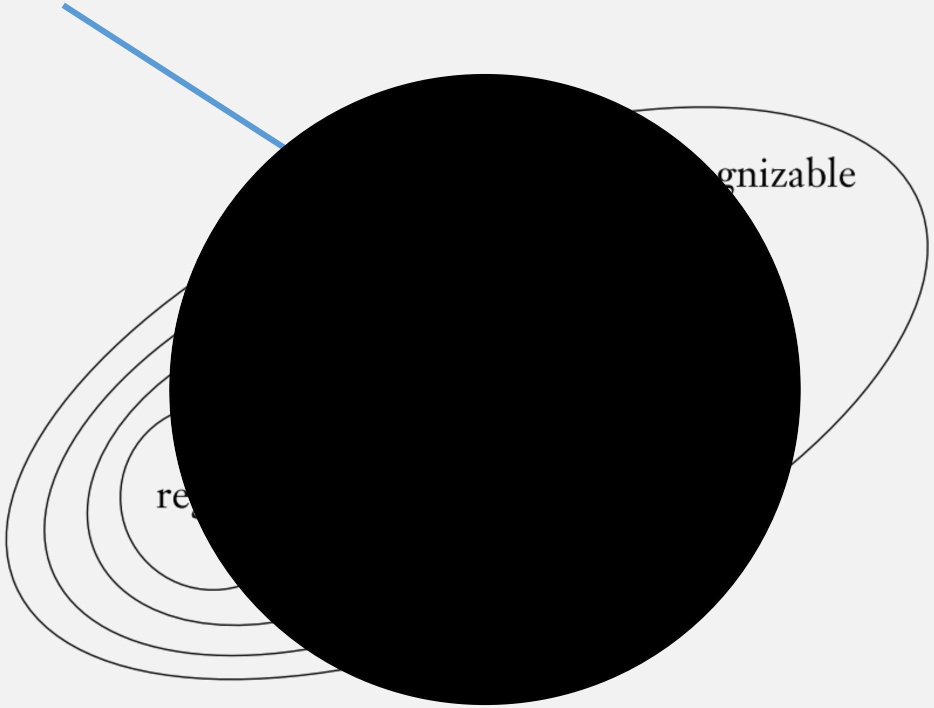
Decidability

Monday, April 8, 2024



Announcements

- HW 7 extended
  - ~~Due Mon 4/8 12pm noon~~
  - Due Wed 4/10 12pm noon
- HW 8 out
  - Due Wed 4/17 12pm noon
- No class Mon 4/15 (Patriots Day)



<u>Quiz Preview (after class)</u>

- What are the required parts of a **decider** TM definition?

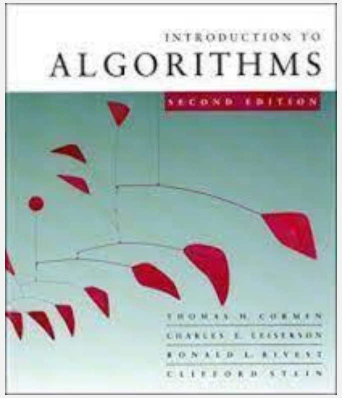
Previously: Turing Machines and Algorithms

- **Turing Machines** can express more "computation" (than other prev machines)
  - Analogy: a **TM models** a (`Python`, `Java`) **program** (`function`)!

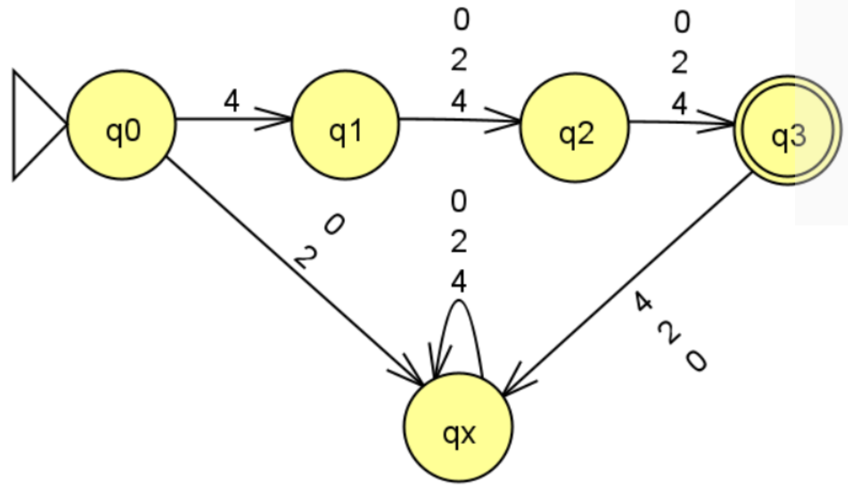
- 2 classes of Turing Machines
  - **Recognizers**: may loop forever
  - **Deciders**: always halt

Today

- **Deciders = Algorithms**
  - I.e., an **algorithm** is a program that (for any input) always **halts**



Flashback: HW 1, Problem 1



1. Come up with 2 strings that are accepted by the DFA. These strings are said to be in the **language** recognized by the DFA.

2. Come up with 2 strings that are not accepted (rejected) by the DFA. These strings are not in the language recognized by the DFA.

3. Come up with a formal description for this DFA.

   Recall that a DFA's formal description is a tuple of five components, e.g. $M = (Q, \Sigma, \delta, q_{start}, F)$.

   You may assume that the alphabet contains only the symbols from the diagram.

4. Then for each of the following, say whether the computation represents an **accepting computation** or not (make sure to review the definition of an accepting computation). If the answer is no, explain why not:

Figuring out this HW problem (about a **DFA's computation**) … <u>is itself</u> (meta) **computation!**

What "kind" of ~~computation~~ **language** is it?

Could you write a <u>program</u> (<u>function</u>) to compute it?

A function: `DFAaccepts(B,w)` returns TRUE if DFA **B** accepts string `w`

1) Define "current" state $q_{current}$ = start state q_0
2) For each input char a_i … in w
   a) Define $q_{next} = \delta_B(q_{current}, a_i)$
   b) Set $q_{current} = q_{next}$
3) Return TRUE if $q_{current}$ is an accept state (of **B**)

Your task: "compute" how a DFA computes

This is "computing": whether we have **accepting computation** $\hat{\delta}(q_0, w) \in F$!!

The language of **`DFAaccepts`**

$$A_{\mathsf{DFA}} = \{\langle B, w\rangle \mid B \text{ is a } \mathsf{DFA} \text{ that accepts input string } w\}$$

How is this language a set of strings???

A function: **`DFAaccepts(B,w)`** returns TRUE if DFA **B** accepts string **`w`**

Interlude: Encoding Things into Strings

Definition: A language's elements / (Turing) machine's input is always a **string**

Problem: We sometimes want TM's (program's) input to be "something else" ...

- **set, graph, DFA, ...?**

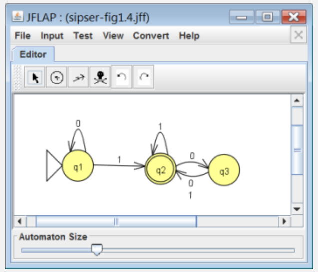
Solution: allow **encoding** "other kinds of input" as a string

Notation: <SOMETHING> = string **encoding** for SOMETHING

- A tuple combines multiple encodings, e.g., <*B*, *w*> (from prev slide)

Details don't matter! (In this class) **Just assume it's possible**

Example: Possible string encoding for a DFA?




```
<automaton>
  <!--The list of states.-->
  <state name="q1"><initial/></state>
  <state name="q2"><final/></state>
  <state name="q3"></state>
  <!--The list of transitions.-->
  <transition>
    <from>0</from>
    <to>0</to>
    <read>0</read>
  </transition>
  <transition>
    <from>1</from>
```


Or:

$$(Q, \Sigma, \delta, q_0, F)$$

(written as string)

Interlude: High-Level TMs and Encodings

A high-level TM description:

1. Needs to say the **type** of its input
   - E.g., graph, DFA, etc.

$M =$ "On input $\langle B, w\rangle$, where B is a DFA and w is a string:

2. Doesn't need to say how input string is encoded

e.g.,

Definition of TM M can use: $B = (Q, \Sigma, \delta, q_0, F)$

- Assume 1: input is a valid encoding
  - Invalid encodings implicitly rejected

Details don't matter! (In this class) **Just assume it's possible**

- Assume 2: TM knows how to parse and extract parts of input

`DFAaccepts` as a TM recognizing A_{DFA}

<u>Remember</u>:
TM ~ program (function)
Creating TM ~ programming

$$A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$$

Previously

A function: `DFAaccepts(B,w)`
returns TRUE if DFA **B** accepts string `w`

1) Define "current" state q_{current} = start state q_0
2) For each input char a_i … in w
   a) Define $q_{\text{next}} = \delta(q_{\text{current}}, a_i)$
   b) Set $q_{\text{current}} = q_{\text{next}}$
3) Return TRUE if q_{current} is an accept state

TM M_{DFA} =

"On inp[ut] $\langle B, w\rangle$, where B is a DFA and w is a string:

Definition of TM M can use: $B = (Q, \Sigma, \delta, q_0, F)$

1) Define "current" state q_{current} = start state q_0
2) For each input char a_i … in w
   a) Define $q_{\text{next}} = \delta(q_{\text{current}}, a_i)$
   b) Set $q_{\text{current}} = q_{\text{next}}$
3) **Accept** if q_{current} is an accept state in F

The language of **DFAaccepts**

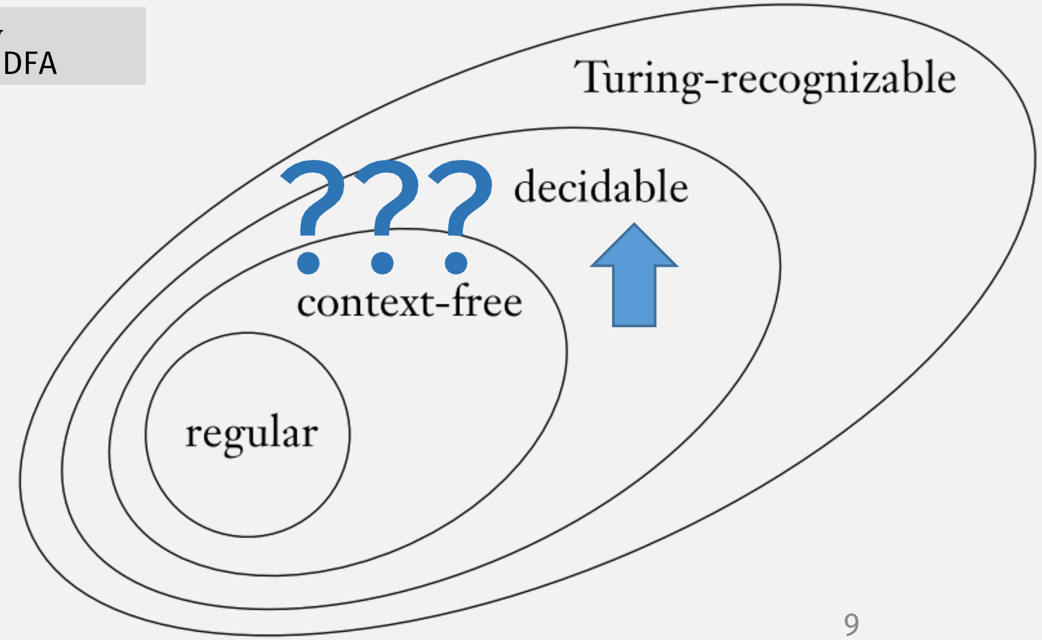
What "kind" of ~~computation~~ language is it?

$$A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$$

TM M_{DFA}

- A_{DFA} has a Turing machine
- Is the TM a **decider** or **recognizer**?
  - I.e., is it an **algorithm**?
- To show it's an algo, need to <u>prove</u>:

A_{DFA} is a decidable language



How to prove that a language is decidable?

How to prove that a language is decidable?

Statements	**Justifications**
1. If a **decider** decides a lang L, then L is a **decidable** lang	1. Definition of **decidable** langs
2. Define **decider** M = On input w ... , M **decides** L	2. See M def, and Equiv. Table
3. L is a **decidable** language	3. By statements #1 and #2

Key step

How to Design Deciders

- A **Decider** is a TM …
  - See previous slides on how to:
    - write a **high-level TM description**
    - Express **encoded** input strings
  - E.g., M = On input <B, w>, where B is a DFA and w is a string: …

- A **Decider** is a TM … that must always **halt**
  - Can only **accept** or **reject**
  - Cannot go into an infinite loop

- So a **Decider** definition must include an extra **termination argument**:
  - Explains how every step in the TM halts
  - (Pay special attention to loops)

- Remember our analogy: TMs ~ Programs … so *Creating* a TM ~ Programm*ing*
  - To design a TM, think of how to write a program (function) that does what you want

Thm: A_{DFA} is a decidable language

$A_{\mathsf{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$

Decider input must match strings in the language!

Key step

Decider for A_{DFA} :

$M =$ "On input $\langle B, w\rangle$, where B is a DFA and w is a string:

1. Simulate B on input w.
2. If the simulation ends in an accept state, *accept*. If it ends in a nonaccepting state, *reject*."

"Calling" the DFA (with an input argument)

Where "Simulate" =

- Define "current" state q_{current} = start state q_0
- For each input char x in w ...
  - Define $q_{\text{next}} = \delta(q_{\text{current}}, x)$
  - Set $q_{\text{current}} = q_{\text{next}}$

Remember:
TM ~ program
Creating TM ~ programming

Thm: A_{DFA} is a decidable language

$A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$

Decider for A_{DFA} :

NOTE: A TM must declare "function" parameters and types … (don't forget it)

$M =$ Undeclared parameters can't be used! (This TM is now invalid because *B, w* are undefined!)

1. Simulate B on input w. ← … which can be used (properly!) in the TM description
2. If the simulation ends in an accept state, *accept*. If it ends in a nonaccepting state, *reject*."

Termination Argument

Thm: A_{DFA} is a decidable language

$A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$

Decider for A_{DFA} :

$M =$ "On input $\langle B, w\rangle$, where B is a DFA and w is a string:

1. Simulate B on input w.
2. If the simulation ends in an accept state, *accept*. If it ends in a nonaccepting state, *reject*."

Where "Simulate" =

- Define "current" state q_{current} = start state q_0
- For each input char x in w …
  - Define $q_{\text{next}} = \delta(q_{\text{current}}, x)$
  - Set $q_{\text{current}} = q_{\text{next}}$

Termination Argument: Step #1 always halts because the simulation <u>input is always finite</u>, so the <u>loop</u> has <u>finite iterations</u> and always halts

Deciders must have a **termination argument:**
Explains how <u>every step</u> in the TM halts (we typically only care about loops)

Thm: A_{DFA} is a decidable language

$A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$

Decider for A_{DFA} :

$M =$ "On input $\langle B, w\rangle$, where B is a DFA and w is a string:

1. Simulate B on input w.
2. If the simulation ends in an accept state, *accept*. If it ends in a nonaccepting state, *reject*."

Termination Argument: **Step #2 always halts because we are checking only the state of the result** (there's no loop)

Deciders must have a **termination argument:**
Explains how *every step* in the TM halts (we typically only care about loops)

Correctness / Equivalence Table

Thm: A_{DFA} is a decidable language

$A_{\text{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$

Decider for A_{DFA} :

M = "On input $\langle B, w\rangle$, where B is a DFA and w is a string:

1. Simulate B on input w.
2. If the simulation ends in an accept state, *accept*. If it ends in a nonaccepting state, *reject*."

(New for TMs) "called" machine column(s)

"Actual" behavior

"Expected" behavior

Example Str	B on input w?	M?	In A_{DFA} lang?
$<D_1, w_1>$	Accept	Accept	Yes
$<D_2, w_2>$	Reject	Reject	No

Let:
- D_1 = DFA, accepts w_1
- D_2 = DFA, rejects w_2

(typically only needed when called machine could loop)

Columns must match!

A good set of examples needs <u>some Yes</u>'s and <u>some No</u>'s

This is what a "Equivalence Table" justification should look like!

Thm: A_{NFA} is a decidable language

$A_{\text{NFA}} = \{\langle B, w\rangle \mid B \text{ is an NFA that accepts input string } w\}$

Decider for A_{NFA} :

Flashback: NFA→DFA

Have: $N = (Q, \Sigma, \delta, q_0, F)$

Want to: construct a DFA $M = (Q', \Sigma, \delta', {q_0}', F')$

1. $Q' = \mathcal{P}(Q)$.

2. For $R \in Q'$ and $a \in \Sigma$,

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a)$$

3. ${q_0}' = \{q_0\}$

4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$

This conversion is computation!

So it can be computed by a (**decider?**) Turing Machine

Turing Machine NFA→DFA

New TM Variation!
Doesn't accept or reject,
Just writes "output" to tape

TM NFA→DFA = On input <N>, where N is an NFA and $N = (Q, \Sigma, \delta, q_0, F)$

1. Write to the tape: DFA $M = (Q', \Sigma, \delta', q_0', F')$

Where: $Q' = \mathcal{P}(Q)$.

For $R \in Q'$ and $a \in \Sigma$,

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a)$$

$$q_0' = \{q_0\}$$

$$F' = \{R \in Q' | \ R \text{ contains an accept state of } N\}$$

Why is this guaranteed to halt?

Because **a DFA description has only finite parts** (finite states, finite transitions, etc)

Thm: A_{NFA} is a decidable language

$A_{\mathsf{NFA}} = \{\langle B, w\rangle \mid B \text{ is an NFA that accepts input string } w\}$

Remember:
TM ~ program
Creating TM ~ programming
Previous theorems ~ library

Decider for A_{NFA}:

"Calling" another TM. Must give correct arg type!

$N =$ "On input $\langle B, w\rangle$, where B is an NFA and w is a string:

1. Convert NFA B to an equivalent DFA C, using the procedure NFA→DFA
2. Run TM M on input $\langle C, w\rangle$. (M is the A_{DFA} decider from prev slide)
3. If M accepts, *accept*; otherwise, *reject*."

New capability: TM can check tape of another TM after calling it

Termination argument: **This is a decider** (i.e., **it always halts**) **because:**

- **Step 1 always halts bc there's a finite number of states in an NFA**
- **Step 2 always halts because M is a decider**

How to Design Deciders, Part 2

Hint:

- **Previous theorems are a "library" of reusable TMs**
- **When creating a TM, try to use this "library" to help you**
  - Just like libraries are useful when programming!
- E.g., **"Library" for DFAs:**
  - **NFA→DFA, RegExpr→NFA**
  - **Union operation, intersect, star, decode, reverse**
  - **Deciders for:** A_{DFA}, A_{NFA}, A_{REX}, …

Thm: A_{REX} is a decidable language

$A_{\mathsf{REX}} = \{\langle R, w\rangle |\ R \text{ is a regular expression that generates string } w\}$

Decider:

NOTE: A TM must declare “function” parameters and types ... (don’t forget it)

$P =$ “On input $\langle R, w\rangle$, where R is a regular expression and w is a string:

1. Convert regular expression R to an equivalent NFA A by using the procedure `RegExpr→NFA`

... which can be used (properly!) in the TM description

Remember:
TMs ~ programs
Creating TM ~ programming
Previous theorems ~ library

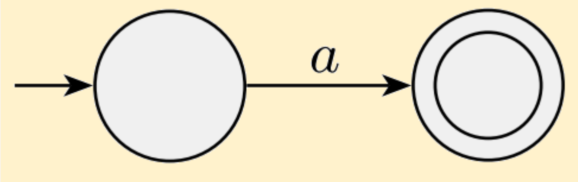
Flashback: RegExpr→NFA

Does this conversion always halt, and why?

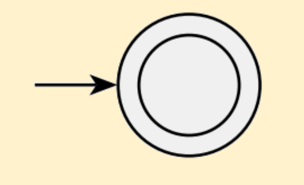
… so guaranteed to always reach base case(s)

R is a ***regular expression*** if R is

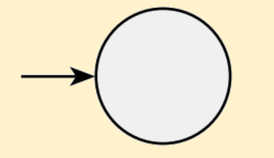
1. a for some a in the alphabet Σ,



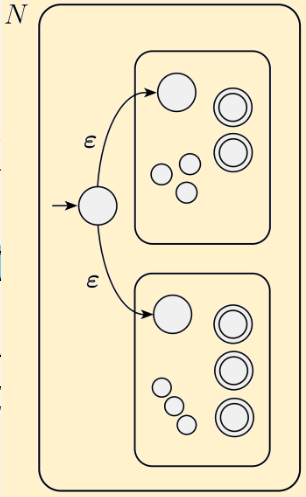
2. ε,

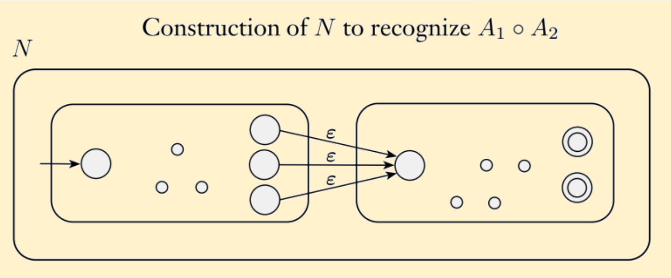


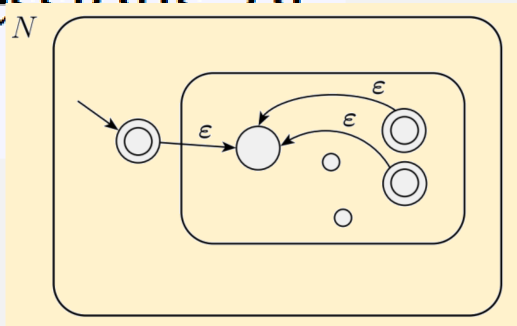
3. $\emptyset$,



4. $(R_1 \cup R_2)$, where R_1 and R_2 a … e
5. $(R_1 \circ R_2)$, where R_1 and R_2 ar … expressions, or
6. (R_1^*), where R_1 is a regular exp







Yes, because recursive call only happens on "smaller" regular expressions …

Thm: A_{REX} is a decidable language

$A_{\mathsf{REX}} = \{\langle R, w\rangle |\ R$ is a regular expression that generates string $w\}$

Decider:

$P =$ "On input $\langle R, w\rangle$, where R is a regular expression and w is a string:

1. Convert regular expression R to an equivalent NFA A by using the procedure **RegExpr→NFA**
2. Run TM N on input $\langle A, w\rangle$. (from prev slide)
3. If N accepts, *accept*; if N rejects, *reject*."

When "calling" another TM, must give proper arguments!

Termination Argument: **This is a decider because:**

- Step 1: always halts because converting a reg expr to NFA is done recursively, where the reg expr gets smaller at each step, eventually reaching the base case
- Step 2: always halts because N is a decider

Decidable Languages About DFAs

- $A_{\text{DFA}} = \{\langle B, w\rangle |\ B \text{ is a DFA that accepts input string}$
  - Decider TM: implements B DFA's extended δ fn

Remember:
TMs ~ programs
Creating TM ~ programming
Previous theorems ~ library

- $A_{\text{NFA}} = \{\langle B, w\rangle |\ B \text{ is an NFA that accepts input string } w\}$
  - Decider TM: uses **NFA→DFA** algorithm + A_{DFA} decider

- $A_{\text{REX}} = \{\langle R, w\rangle |\ R \text{ is a regular expression that generates string } w\}$
  - Decider TM: uses **RegExpr→NFA** algorithm + A_{NFA} decider

Flashback: Why Study Algorithms About Computing

To predict what programs will do

(without running them!)

Not possible for all programs! But …

???

Predicting What Some Programs Will Do …

What if: look at simpler computation models
… like DFAs and regular languages!

Thm: E_{DFA} is a decidable language

$$E_{\mathsf{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$$

E_{DFA} is a language … of DFA descriptions, i.e., $(Q, \Sigma, \delta, q_0, F)$ …

… where **the language of** each DFA … **must be { }**, i.e., **DFA accepts no strings**

Is there a decider that accepts/rejects DFA descriptions …

… by predicting something about the DFA's language (by analyzing its description)

Key question we are studying:
Compute (predict) something about the runtime computation of a **program**, by **analyzing *only*** its source code?

Analogy
DFA's description : a program's source code **::**
DFA's language : a program's runtime computation

Important: **don't confuse the different languages here!**

Thm: E_{DFA} is a decidable language

$E_{\mathsf{DFA}} = \{\langle A \rangle |\ A \text{ is a DFA and } L(A) = \emptyset\}$

Decider:

T = "On input $\langle A \rangle$, where A is a DFA:

1. Mark the start state of A.

2. Repeat until no new states get marked:

3. Mark any state that has a transition coming into it from any state that is already marked.

4. If no accept state is marked, *accept*; otherwise, *reject*."

If loop marks at least 1 state on each iteration, then it eventually terminates because there are finite states; else loop terminates

Termination argument?

I.e., this is a "reachability" algorithm …

… check if accept states are "reachable" from start state

Note: TM T does not "run" the DFA!

… it computes something about the DFA's language (runtime computation) by analyzing it's description (source code)

Thm: EQ_{DFA} is a decidable language

$$EQ_{\text{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$$

I.e., Can we compute whether two DFAs are "equivalent"?

Replacing "**DFA**" with "**program**" =
A "**holy grail**" **of computer science**!

Thm: EQ_{DFA} is a decidable language

$$EQ_{\text{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$$

A Naïve Attempt (assume alphabet **{a}**):

1. Simulate:
   - A **with input a,** and
   - B **with input a**
   - **Reject** if results are different, else …
2. Simulate :
   - A **with input aa,** and
   - B **with input aa**
   - **Reject** if results are different, else …

- …

This might not terminate!
(Hence it's not a decider)

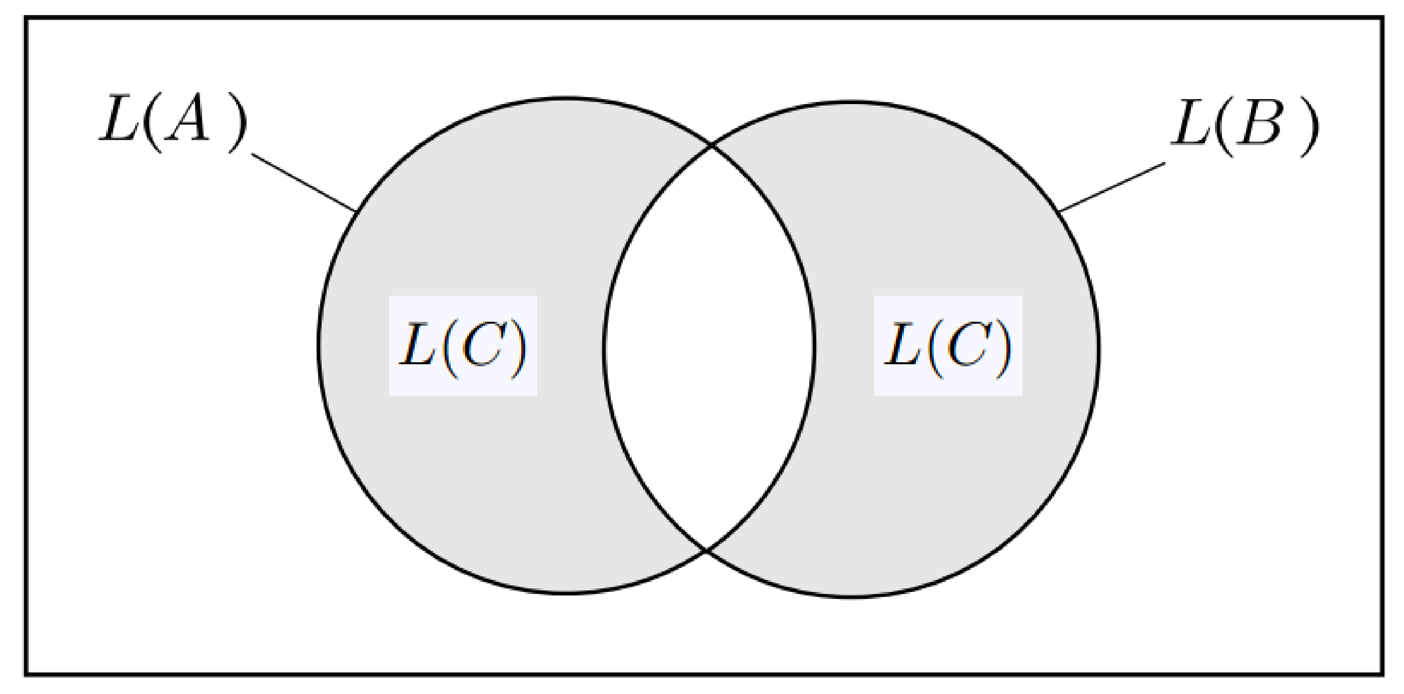
Can we compute this <u>without</u> running the DFAs?

Thm: EQ_{DFA} is a decidable language

$$EQ_{\mathsf{DFA}} = \{\langle A, B\rangle |\ A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$$

Trick: Use Symmetric Difference

Symmetric Difference



$$L(C) = \left(L(A) \cap \overline{L(B)}\right) \cup \left(\overline{L(A)} \cap L(B)\right)$$

$$L(C) = \emptyset \text{ iff } L(A) = L(B)$$

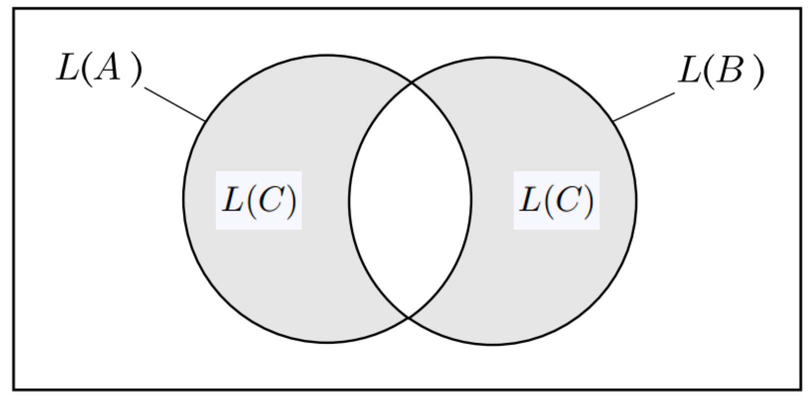
Thm: EQ_{DFA} is a decidable language

$$EQ_{\mathsf{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$$

NOTE, This only works because:
regular langs closed under **negation**,
i.e., set complement, **union** and **intersection**

Construct **decider** using 2 parts:

1. Symmetric Difference algo: $L(C) = \left(L(A) \cap \overline{L(B)}\right) \cup \left(\overline{L(A)} \cap L(B)\right)$
   - Construct C = Union, intersection, negation of machines A and B
2. Decider T (from "library") for: $E_{\mathsf{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$
   - Because $L(C) = \emptyset$ iff $L(A) = L(B)$



Thm: EQ_{DFA} is a decidable language

$EQ_{\mathsf{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$

Construct **decider** using 2 parts:

1. Symmetric Difference algo: $L(C) = \left(L(A) \cap \overline{L(B)}\right) \cup \left(\overline{L(A)} \cap L(B)\right)$
   - Construct C = Union, intersection, negation of machines A and B
2. Decider T (from "library") for: $E_{\mathsf{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$
   - Because $L(C) = \emptyset \text{ iff } L(A) = L(B)$

F = "On input $\langle A, B\rangle$, where A and B are DFAs:

1. Construct DFA C as described.
2. Run TM T deciding E_{DFA} on input $\langle C\rangle$.
3. If T accepts, *accept*. If T rejects, *reject*."

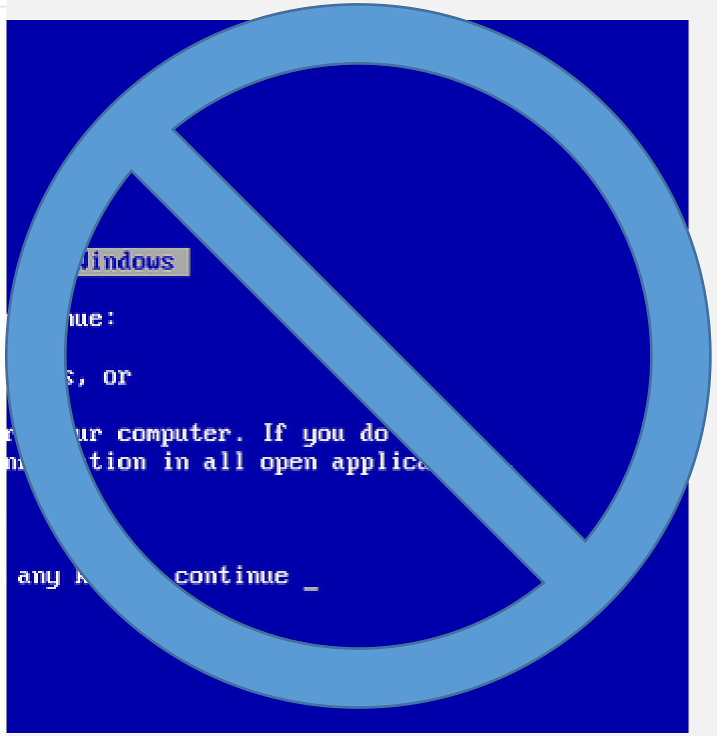
Termination argument?

Predicting What <u>Some</u> Programs Will Do …

microsoft.com/en-us/research/project/slam/

SLAM is a project for checking that software satisfies critical behavioral properties of the interfaces it uses and to aid software engineers in designing interfaces and software that ensure reliable and correct functioning. Static Driver Verifier is a tool in the Windows Driver Development Kit that uses the SLAM verification engine.

"Things like even software verification, this has been the Holy Grail of computer science for many decades but now in some very key areas, for example, driver verification we're building tools that can do actual proof about the software and how it works in order to guarantee the reliability." **Bill Gates, April 18, 2002. Keynote address at WinHec 2002**



Static Driver Verifier Research Platform README

Overview of Static Driver Verifier Research Platform

Static Driver Verifier (SDV) is a compile-time static verification
Research Platform (SDVRP) is an extension to SDV that allows

- Support additional frameworks (or APIs) and write custo
- Experiment with the model checking step.

Model checking

From Wikipedia, the free encyclopedia

In computer science, **model checking** or **property checking** is a method for checking whether a finite-state model of a system meets a given specification (also known as correctness). This is typically

Its "language"

Summary: Algorithms About Regular Langs

- $A_{\mathsf{DFA}} = \{\langle B, w\rangle \mid B \text{ is a DFA that accepts input string } w\}$
  - **Decider**: Simulates DFA by implementing extended δ function
- $A_{\mathsf{NFA}} = \{\langle B, w\rangle \mid B \text{ is an NFA that accepts input string } w\}$
  - **Decider**: Uses **NFA→DFA** decider + A_{DFA} decider
- $A_{\mathsf{REX}} = \{\langle R, w\rangle \mid R \text{ is a regular expression that generates string } w\}$
  - **Decider**: Uses **RegExpr→NFA** decider + A_{NFA} decider
- $E_{\mathsf{DFA}} = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \emptyset\}$
  - **Decider**: Reachability algorithm

Lang of the DFA

- $EQ_{\mathsf{DFA}} = \{\langle A, B\rangle \mid A \text{ and } B \text{ are DFAs and } L(A) = L(B)\}$

  - **Decider**: Uses complement and intersection closure construction + E_{DFA} decider

Remember:
TMs ~ programs
Creating TM ~ programming
Previous theorems ~ library

Next: Algorithms (Decider TMs) for CFLs?

- What can we predict about CFGs or PDAs?