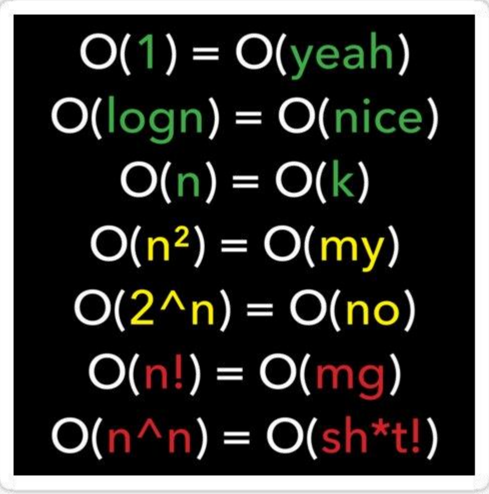
UMB CS420

Polynomial Time (P)

Monday, May 6, 2024



Announcements

- HW 11
  - Due Wed 5/8 12pm noon

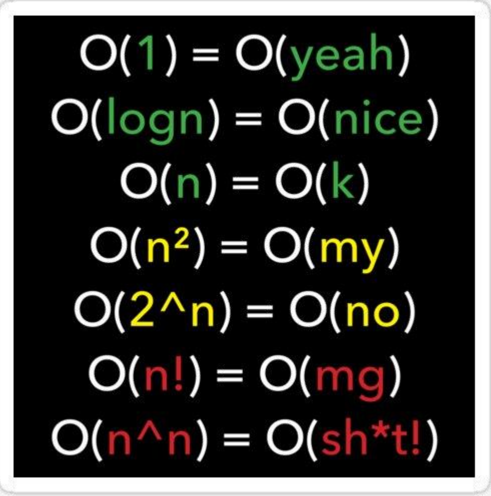
- HW 12
  - Release Wed 5/8 12pm noon
  - Due Wed 5/15 12pm noon (no late days, no exceptions)

Quiz Preview

Q1 The time complexity class P represents what kind of problems ?
1 Point

(select all that apply)

- [ ] realistically solvable problems
- [ ] tractable problems
- [ ] problems that have a polynomial time algorithm
- [ ] languages decided by Turing-machines that run in a worst case polynomial number of steps



Last Time: Time Complexity

Running Time or **Time Complexity** is a property of decider TMs (algorithms)

Let M be a deterministic Turing machine that halts on all inputs. The ***running time*** or ***time complexity*** of M is the function $f: \mathcal{N} \longrightarrow \mathcal{N}$, where $f(n)$ is the maximum number of steps that M uses on any input of length n. If $f(n)$ is the running time of M, we say that M runs in time $f(n)$ and that M is an $f(n)$ time Turing machine. Customarily we use n to represent the length of the input.

Depends on size of input

Worst case

Last Time: Time Complexity Classes

Big-O = asymptotic upper bound,
i.e., "only care about large n"

Let $t\colon \mathcal{N} \longrightarrow \mathcal{R}^+$ be a function. Define the ***time complexity class***, **TIME($t(n)$)**, to be the collection of all languages that are decidable by an $O(t(n))$ time Turing machine.

Remember:
- TMs: have **a time complexity** (i.e., a running time),
- languages: are in **a time complexity class**

The **time complexity class** of a language is determined by the **time complexity** (running time) of its deciding TM

A language can have multiple deciding TMs, so could be in multiple **time complexity classes**

The Polynomial Time Complexity Class (**P**)

P is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$P = \bigcup_k \text{TIME}(n^k).$$

- Corresponds to “realistically” solvable problems:
  - Problems in **P**
    - = “**solvable**” or “**tractable**”
  - Problems outside **P**
    - = “**unsolvable**” or “**intractable**”

“Unsolvable” Problems

Mathematicians are weird.

- Unsolvable problems (those outside **P**):
  - usually only have “brute force” solutions
  - i.e., “try all possible inputs”
  - “unsolvable” applies only to large ***n***

Brute-force attack

From Wikipedia, the free encyclopedia

In cryptography, a **brute-force attack** consists of an attacker submitting many passwords or passphrases with the hope of eventually guessing a combination correctly. The attacker systematically checks all possible passwords and passphrases until the correct one is found. Alternatively, the attacker can attempt to guess the key which is typically created from the password using a key derivation function. This is known as an **exhaustive key search**.

Amount of Time to Crack Passwords

“abcdefg” 7 characters	.29 milliseconds
“abcdefgh” 8 characters	5 hours
“abcdefghi” 9 characters	5 days
“abcdefghij” 10 characters	4 months
“abcdefghijk” 11 characters	1 decade
“abcdefghijkl” 12 characters	2 centuries

In this class, we’re interested in questions like:

today → How to prove something is “**solvable**” (in **P**)?

How to prove something is “**unsolvable**” (not in **P**)?

3 Problems in $\mathbf{P}$

- A Graph Problem:

  $PATH = \{\langle G, s, t\rangle \mid G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$

- A Number Problem:

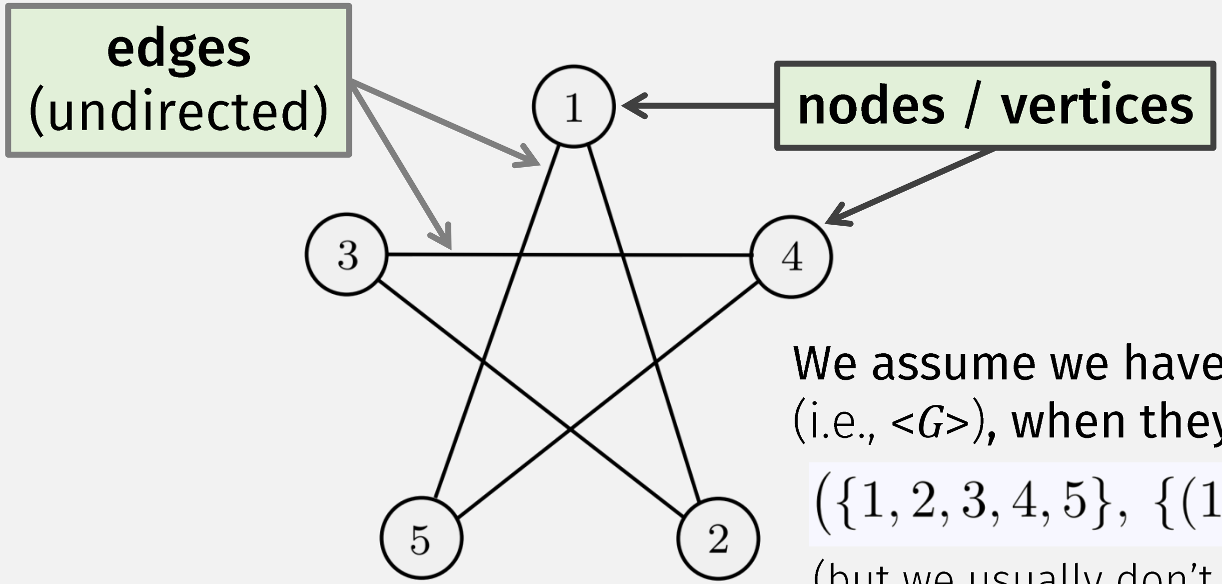
  $RELPRIME = \{\langle x, y\rangle \mid x \text{ and } y \text{ are relatively prime}\}$

- A CFL Problem:

  Every context-free language is a member of P

- To prove that a language is "solvable", i.e., in $\mathbf{P}$ ...
  - ... construct a polynomial time algorithm deciding the language
- (These may also have nonpolynomial, i.e., brute force, algorithms)
  - Check all possible ... paths/numbers/strings ...

Interlude: Graphs (see Sipser Chapter 0)



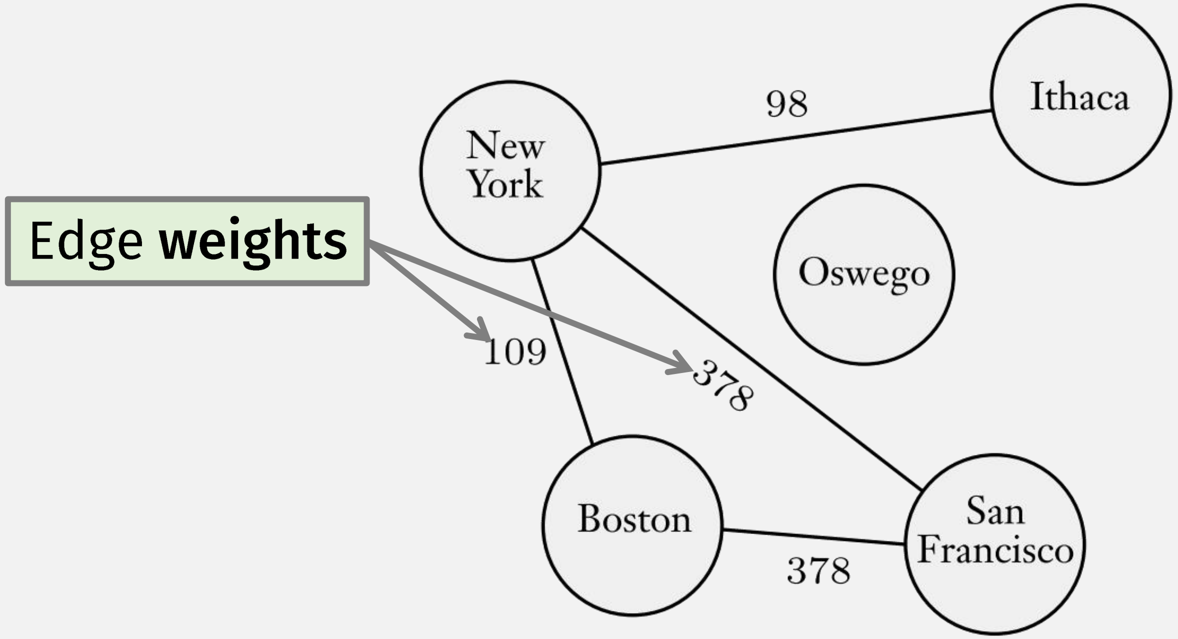
We assume we have *some* string encoding of a graph (i.e., <*G*>), when they are args to TMs, e.g.:

$$(\{1,2,3,4,5\},\ \{(1,2),\ (2,3),\ (3,4),\ (4,5),\ (5,1)\})$$

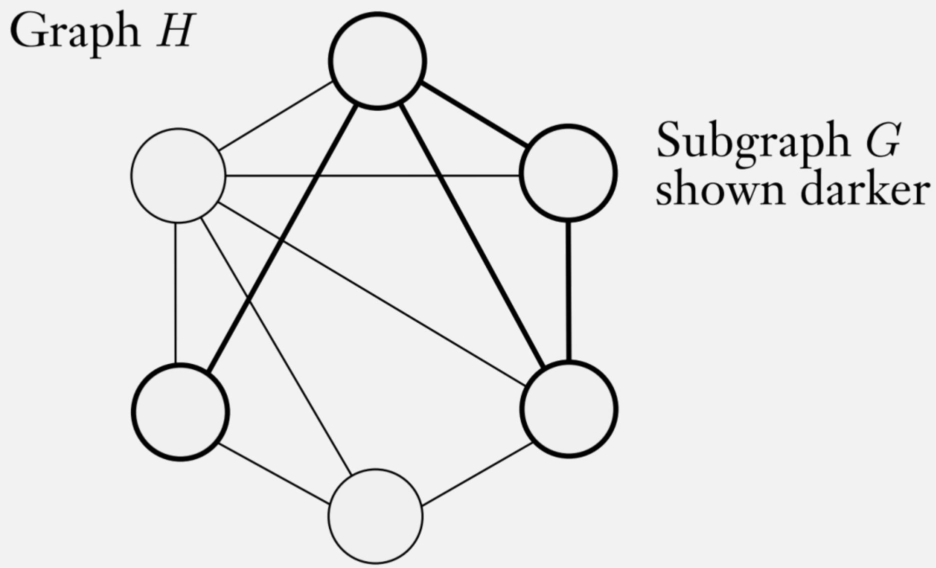
(but we usually don't care about the actual details)

- **Edge** defined by two **nodes** (order doesn't matter)
- Formally, a **graph** = a pair (V, E)
  - Where V = a set of nodes, E = a set of edges

Interlude: Weighted Graphs

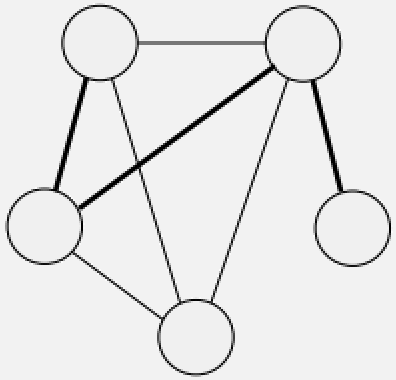


Interlude: Subgraphs

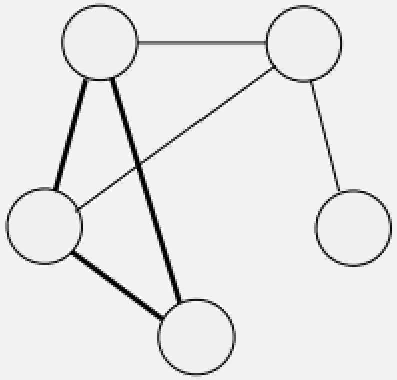


Interlude: Paths and other Graph Things

- **Path**
  - A sequence of nodes connected by edges

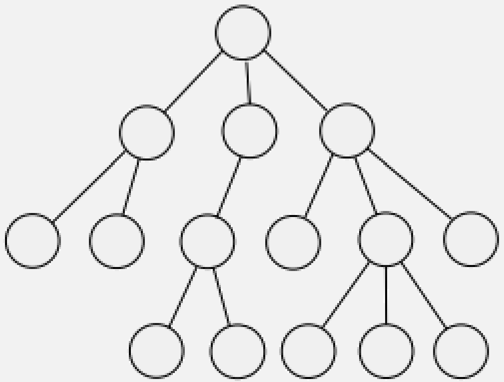


- **Cycle**
  - A path that starts/ends at the same node

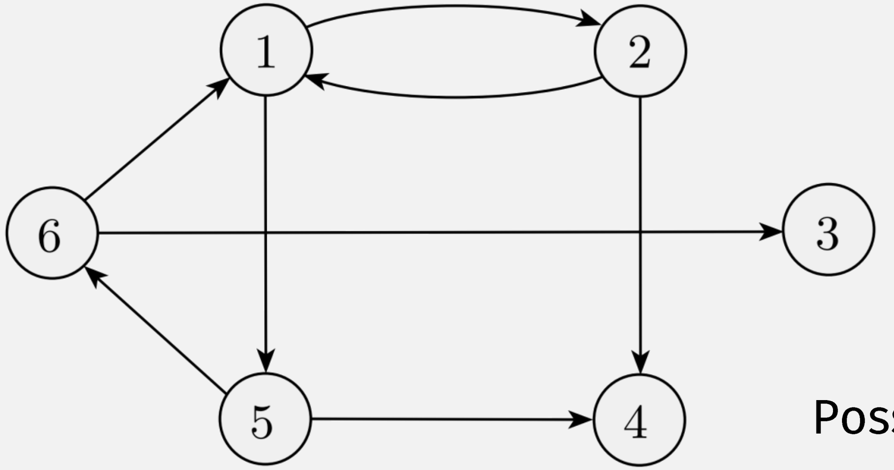


- **Connected graph**
  - Every two nodes has a path

- **Tree**
  - A connected graph with no cycles



Interlude: Directed Graphs



Possible string encoding given to TMs:

$(\{1,2,3,4,5,6\}, \{(1,2), (1,5), (2,1), (2,4), (5,4), (5,6), (6,1), (6,3)\})$

Each pair of nodes included twice

- **Directed graph** = (V, E)
  - V = set of nodes, E = set of edges
- An **edge** is a pair of nodes (u,v), *order now matters*
  - u = "from" node, v = "to" node
- "degree" of a node: number of edges connected to the node
  - Nodes in a directed graph have both indegree and outdegree

Interlude: Graph Encodings

$(\{1,2,3,4,5\},\ \{(1,2),\ (2,3),\ (3,4),\ (4,5),\ (5,1)\})$

- For graph algorithms, "length of input" n usually = # of vertices
  - (Not number of chars in the encoding)
- So given graph $G = (V, E)$, $n = |V|$
- Max edges?
  - $= O(|V|^2) = O(n^2)$
- So if a set of graphs (call it lang L) is decided by a TM where
  - # steps of the TM = polynomial in the # of vertices
    Or polynomial in the # of edges
  - Then L is in **P**

3 Problems in **P**

- A Graph Problem:

  $PATH = \{\langle G, s, t\rangle \mid G$ is a directed graph that has a directed path from s to $t\}$

- A Number Problem:

  $RELPRIME = \{\langle x, y\rangle \mid x$ and y are relatively prime$\}$

- A CFL Problem:

  Every context-free language is a member of P

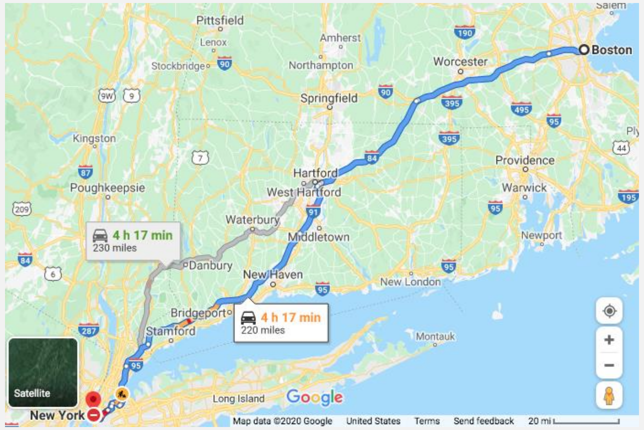
P is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$P = \bigcup_k \text{TIME}(n^k).$$

A Graph Theorem: $PATH \in \text{P}$

$$PATH = \{\langle G, s, t\rangle \mid G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$$

(A **path** is a sequence of nodes connected by edges)



- To prove that a language is in **P** …
- … we must construct a polynomial time algorithm deciding the lang
- A non-polynomial (i.e., ”brute force”) algorithm:
  - check all possible combination of all vertices,
  - see if any connect s to t
  - If n = # vertices, then # paths $\approx n^n$ or $n!$ (worse than $2^{O(n)}$)

A Graph Theorem: $PATH \in \mathrm{P}$

$PATH = \{\langle G, s, t\rangle \mid G$ is a directed graph that has a directed path from s to $t\}$

PROOF A polynomial time algorithm M for $PATH$ operates as follows.

M = "On input $\langle G, s, t\rangle$, where G is a directed graph with nodes s and t:

1. Place a mark on node s.

2. Repeat the following until no additional nodes are marked:

3. Scan all the edges of G. If an edge (a, b) is found going from a marked node a to an unmarked node b, mark node b.

4. If t is marked, *accept*. Otherwise, *reject*."

# of steps (worst case) **(*n* = # nodes):**

- Line 1: **1 step**

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- Lines 2-3 (loop):
  - Steps/iteration (line 3): max # steps = max # edges = $O(n^2)$

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(Breadth-first search)

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  - ➢ Total: $O(n^3)$

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- Line 4: **1 step**

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$PATH \in \text{TIME}(\boldsymbol{n}^3)$

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$O(\boldsymbol{n}^3)$

# of steps (worst case) **($\boldsymbol{n}$ = # nodes):**

- Line 1: **1 step**
- Lines 2-3 (loop):
  - Steps/iteration (line 3): max # steps = max # edges = $O(\boldsymbol{n}^2)$
  - # iterations (line 2): loop runs at most $\boldsymbol{n}$ times
  - Total: $O(\boldsymbol{n}^3)$
- Line 4: **1 step**

➢ Total = 1 + 1 + $O(\boldsymbol{n}^3)$ = $O(\boldsymbol{n}^3)$

3 Problems in **P**

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• A CFL Problem:

Every context-free language is a member of P

A Number Theorem: $RELPRIME \in \mathrm{P}$

$RELPRIME = \{\langle x, y\rangle \mid x \text{ and } y \text{ are relatively prime}\}$

- Two numbers are **relatively prime:** if their gcd = 1
  - gcd(x, y) = largest number that divides both x and y
  - E.g., gcd(8, 12) = ??
- Brute force (exponential) algorithm deciding $RELPRIME$:
  - Try all of numbers (up to x or y), see if it can divide both numbers

  Q: Why is this exponential?

  HINT: What is a typical "representation" of numbers?

  A: binary numbers

  (if $x = 2^n$, then trying x numbers is exponential in n, the number of digits)
- A gcd algorithm that runs in polynomial time:
  - Euclid's algorithm

A GCD Algorithm for: $RELPRIME \in \mathrm{P}$

$RELPRIME = \{\langle x, y\rangle \mid x \text{ and } y \text{ are relatively prime}\}$

Modulo
(i.e., **remainder**)

15 mod 8 = 7
17 mod 8 = 1

cuts x (at least) **in half every loop, requires:** **log x** loops

The Euclidean algorithm E is as follows.

E = "On input $\langle x, y\rangle$, where x and y are natural numbers in binary:

1. Repeat until $y = 0$:
2. Assign $x \leftarrow x \bmod y$.
3. Exchange x and y.
4. Output x."

***O*(*n*)**

Each number is cut in half every *other* iteration

<u>Total run time</u> (assume $x > y$): $2\log x = 2\log 2^n =$ ***O*(*n*)**,
where n = number of binary digits in (ie length of) x

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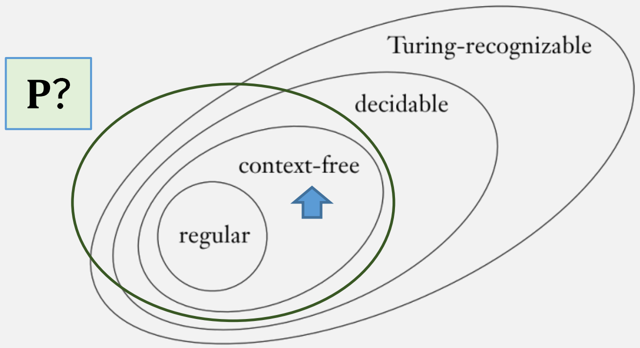

IF-THEN Statement to Prove:

IF a language L is a CFL,
THEN L is in **P**

Review: A Decider for Any CFL

Given any CFL L, with CFG G, the following decider M_G decides L:

$M_G =$ "On input w:
1. Run TM S on input $\langle G, w \rangle$.
2. If this machine accepts, *accept*; if it rejects, *reject*."

S is a decider for: $A_{\text{CFG}} = \{\langle G, w \rangle | \ G \text{ is a CFG that generates string } w\}$

$S =$ "On input $\langle G, w \rangle$, where G is a CFG and w is a string:
1. Convert G to an equivalent grammar in Chomsky normal form.
2. List all derivations with $2n - 1$ steps, where n is the length of w; except if $n = 0$, then instead list all derivations with one step.
3. If any of these derivations generate w, *accept*; if not, *reject*."

M_G is a decider, bc S is a decider

M_G accepts all $w \in L$, for any CFL L (with CFL G)

Therefore, every CFL is decidable

But, is every CFL decidable in poly time?

A Decider for Any CFL: Running Time

Given any CFL L, with CFG G, the following decider M_G decides L:

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$A \to 0A1$
$A \to B$
$B \to \#$

How many different possibilities at *each* derivation step?

$A \Rightarrow 0A1 \Rightarrow$ ...

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Worst case:
$|R|^{2n-1}$ steps = $O(2^n)$
(R = set of rules)

This algorithm runs in exponential time

A CFL Theorem: Every context-free language is a member of P

- Given a CFL, we must construct a decider for it …
- … that runs in polynomial time

Dynamic Programming

- **Keep track of partial solutions,** and **re-use them**
  - Start with smallest and build up
- **For CFG problem,** instead **of re-generating entire string …**
  - **… keep track of <u>substrings</u> generated by each variable**

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3. If any of these derivations generate w, *accept*; if not, *reject*."

This <u>duplicates a lot of work</u> because many strings might have the <u>same beginning derivation steps</u>

CFL Dynamic Programming Example

- Chomsky Grammar G:
  - S → AB | BC
  - A → BA | a
  - B → CC | b
  - C → AB | a
- Example string: **baaba**
- Store every partial string and their generating variables in a table

Substring end char

Substring start char

	b	**a**	**a**	**b**	**a**
b					
a					
a					
b					
a					

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Substring <u>end</u> char

Substring <u>start</u> char

	b	**a**	**a**	**b**	**a**
b	vars generating "b"	vars for "ba"	vars for "baa"	…	
a		vars for "a"	vars for "aa"	vars for "aab"	
a			…		
b					
a					

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Algo:
- For each single char c and var A:
  - If A → c is a rule, add A to table

Substring end char

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	b	**a**	**a**	**b**	**a**
b	B				
a		A,C			
a			A,C		
b				B	
a					A,C

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Algo:
- For each single char c and var A:
  - If A → c is a rule, add A to table
- For each substring s (len > 1):
  - For each split of substring s into x,y:
    - For each rule of shape A → BC:
      - Use table to check if B generates x and C generates y

Substring end char

Substring start char

	b	**a**	**a**	**b**	**a**
b	B				
a		A,C			
a			A,C		
b				B	
a					A,C

CFL Dynamic Programming Example

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  - S → AB | BC
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  - C → AB | a
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- Store every partial string and their ge[nerating]…

Algo:
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- For each substring s:
  - For each split of substring s into x,y:
    - For each rule of shape A → BC:
      - Use table to check if B …

For substring "**ba**", split into "**b**" and "**a**":
- For rule S → AB
  - Does A generate "**b**" and B generate "**a**"?
  - NO
- For rule S → BC
  - Does B generate "**b**" and C generate "**a**"?
  - **YES**
- For rule A → BA
  - Does B generate "**b**" and A generate "**a**"?
  - **YES**
- For rule B → CC
  - Does C generate "**b**" and C generate "**a**"?
  - NO
- For rule C → AB
  - Does A generate "**b**" and B generate "**a**"?
  - NO

Substring end char

Substring start char

	b	**a**	**a**
b	B		
a		A,C	
a			A,C
b			
a			

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Substring end char

Substring start char

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a		A,C	
a			A,C
b			
a			

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  - Does A generate "**b**" and B generate "**a**"?
  - NO
- For rule S → BC
  - Does B generate "**b**" and C generate "**a**"?
  - **YES**
- For rule A → BA
  - Does B generate "**b**" and A generate "**a**"?
  - **YES**
- For rule B → CC
  - Does C generate "**b**" and C generate "**a**"?
  - NO
- For rule C → AB
  - Does A generate "**b**" and B generate "**a**"?
  - NO

CFL Dynamic Programming Example

- Chomsky Grammar G:
  - C
  - B → CC | b
  - C → AB | a
- ing: **baaba**
- partial string and their generating variables in a table

For each:
- char
- var

For each:
- substring
- split of substring
- rule

Algo:
- For each single char c and var A:
  - If A → c is a rule, add A to table
- For each substring s:
  - For each split of substring s into x,y:
    - For each rule of shape A → BC:
      - Use table to check if B generates x and C generates y

For each: char, var …

For each: substring, split, rule …

Substring end char

Substring start char

	b	a	a	b	a
b	B	S,A			S,A,C
a		A,C	B	B	S,A,C
a			A,C	S,C	B
b				B	S,A
a					A,C

If S is here, accept → S,A,C

A CFG Theorem: Every context-free language is a member of P

$D =$ "On input $w = w_1 \cdots w_n$:

1. For $w = \varepsilon$, if $S \to \varepsilon$ is a rule, *accept*; else, *reject*. [[$w = \varepsilon$ case]]
2. For $i = 1$ to n: [[examine each substring of length 1]]
3. For each variable A:
4. Test whether $A \to$ b is a rule, where b $= w_i$.
5. If so, place A in $table(i, i)$.
6. For $l = 2$ to n: [[l is the length of the substring]]
7. For $i = 1$ to $n - l + 1$: [[... substring]]
8. Let $j = i + l - 1$. [[j is the end position of the substring]]
9. For $k = i$ to $j - 1$:
10. For each rule $A \to BC$:
11. If $table(i, k)$ contains B and $table(k + 1, j)$ contains C, put A in $table(i, j)$.
12. If S is in $table(1, n)$, *accept*; else, *reject*."

For each:
- char
- var

$O(n)$ chars

#vars = constant = $O(1)$

$O(1) * O(n) = O(n)$

For each:
- substring
- split of substring
- rule

$O(n)$ diff lengths

$O(n)$ strings of each length

$O(n)$ ways to split a string into two pieces

#vars = constant = $O(1)$

$O(1) * O(n) * O(n) * O(n) = O(n^3)$

<u>Total</u>: $O(n^3)$

(This is also known as the <u>Earley parsing algorithm</u>)

Summary: 3 Problems in $\mathbf{P}$

☑ • A Graph Problem:

$PATH = \{\langle G, s, t\rangle \mid G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$

☑ • A Number Problem:

$RELPRIME = \{\langle x, y\rangle \mid x \text{ and } y \text{ are relatively prime}\}$

☑ • A CFL Problem:

Every context-free language is a member of P

Lecture participation question 5/6

On gradescope