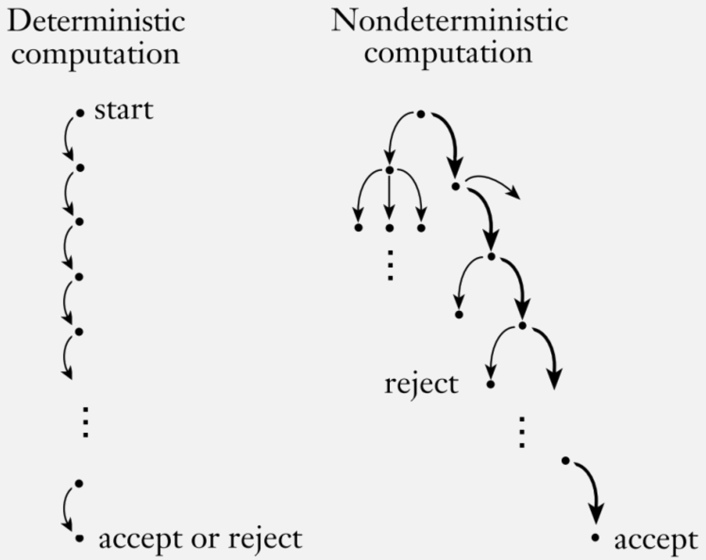
UMB CS622

Nondeterministic Finite Automata (NFAs)

Wednesday September 15, 2021



Announcements

- HW1 due Sun 9/19 11:59pm EST
  - Upload solutions to Gradescope
  - LaTeX is great!
  - Handwritten and scanned/photo is perfectly fine
  - I must be able to read your answers!
    - Illegible solutions will not receive any credit

- Please post HW questions to Piazza
  - Don't email me directly
  - So others can benefit from the discussion, and potentially help out!

- Monday 9/13 lecture video posted

- Welcome new students!
  - Make sure to catch up ASAP

Last Time: Finite State Automaton, a.k.a. DFAs

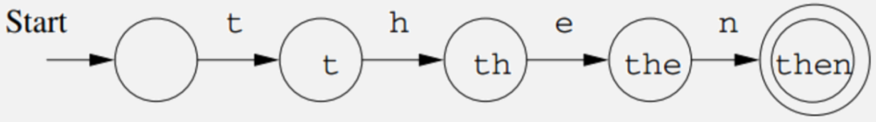
DEFINITION 1.5

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta\colon Q \times \Sigma \longrightarrow Q$ is the ***transition function***,
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

- Key characteristic:
  - Has a **finite** number of states
  - I.e., it's a computer with a finite amount of memory
    - Can't dynamically allocate

- Often used for **text matching**



Combining DFAs?

Password Requirements

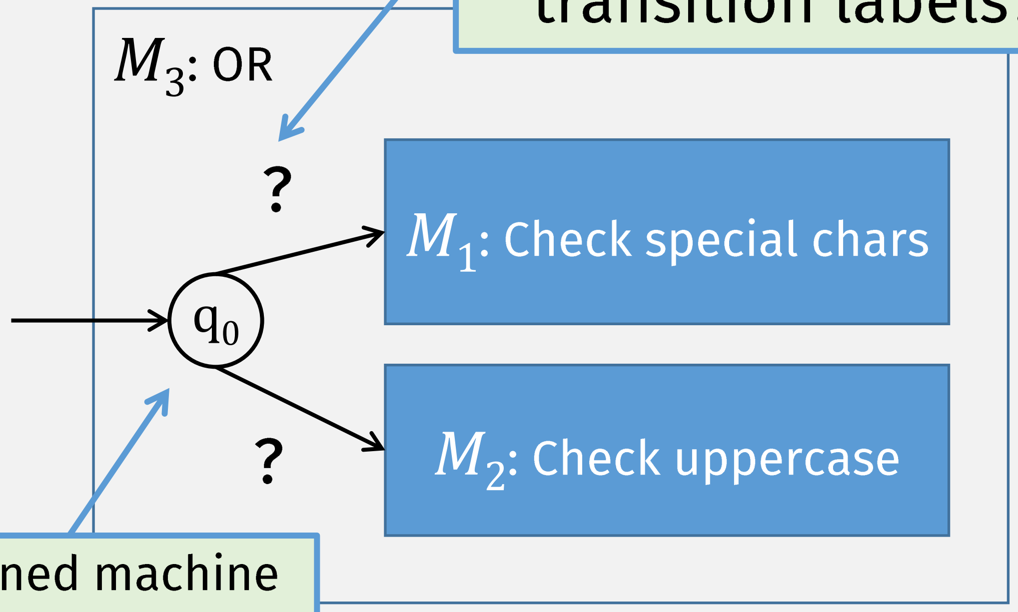
- Passwords must have a minimum length of ten (10) characters - but more is better! ← DFA
- Passwords **must include at least 3** different types of characters:
  - upper-case letters (A-Z) ← DFA
  - lower-case letters (a-z) ← DFA
  - symbols or special characters (%, &, *, $, etc.) ← DFA
  - numbers (0-9) ← DFA
- Passwords cannot contain all or part of your email address ← DFA
- Passwords cannot be re-used ← DFA

s://www.umb.edu/it/password

To match <u>all</u> requirements,
can we combine smaller DFAs?

Combining DFAs

<u>Problem 1</u>:
What should be the transition labels?



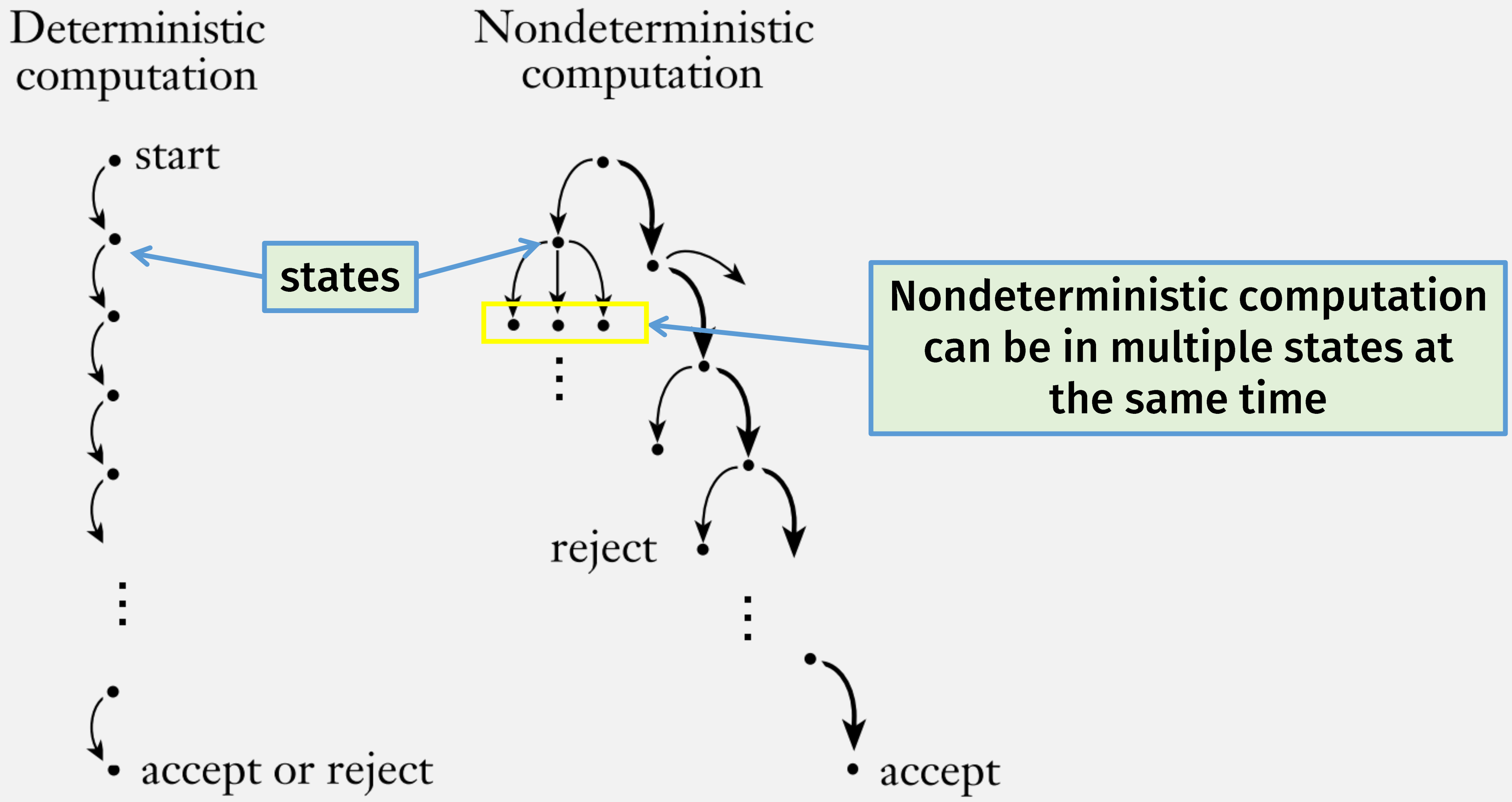
Combined machine adds new start state

<u>Problem 2</u>:
Once we enter one of the machines, can't go back to the other one!

We need a different kind of machine!

<u>Idea:</u> **nondeterminism** allows being in multiple states (i.e., multiple machines) at once!

Nondeterminism



Nondeterministic Finite Automata (NFA)

DEFINITION 1.37

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta\colon Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Compare with DFA:

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

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Difference

Power set, i.e. a transition results in set of states

Power Sets

- A power set is the set of all subsets of a set

- Example: S = {**a**, **b**, **c**}

- Power set of S =
  - {{ }, {**a**}, {**b**}, {**c**}, {**a**, **b**}, {**a**, **c**}, {**b**, **c**}, {**a**, **b**, **c**}}
  - Note: includes the empty set!

Nondeterministic Finite Automata (NFA)

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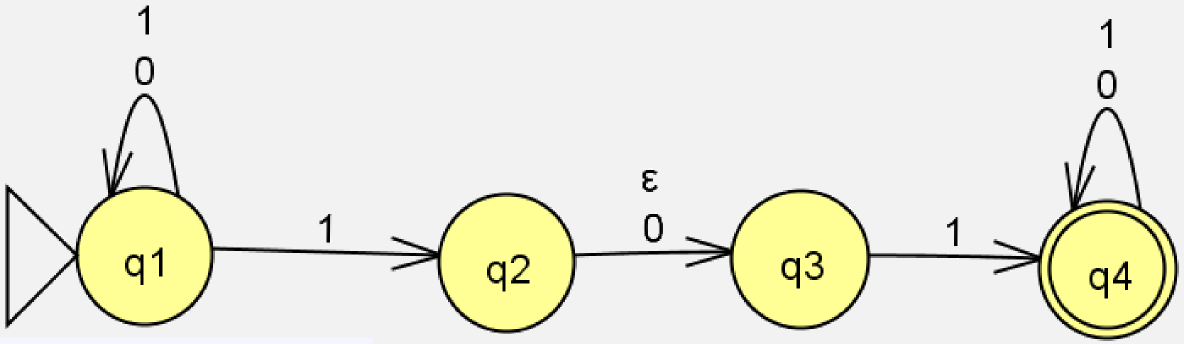
5. $F \subseteq Q$ is the ***set of accept states***.

Transition label can be "empty", i.e., machine can transition without reading input

$$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$$

NFA Example

- Come up with a formal description of the following NFA:



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The formal description of N_1 is $(Q, \Sigma, \delta, q_1, F)$, where

$$\delta : Q \times \Sigma_{\varepsilon} \longrightarrow \mathcal{P}(Q)$$

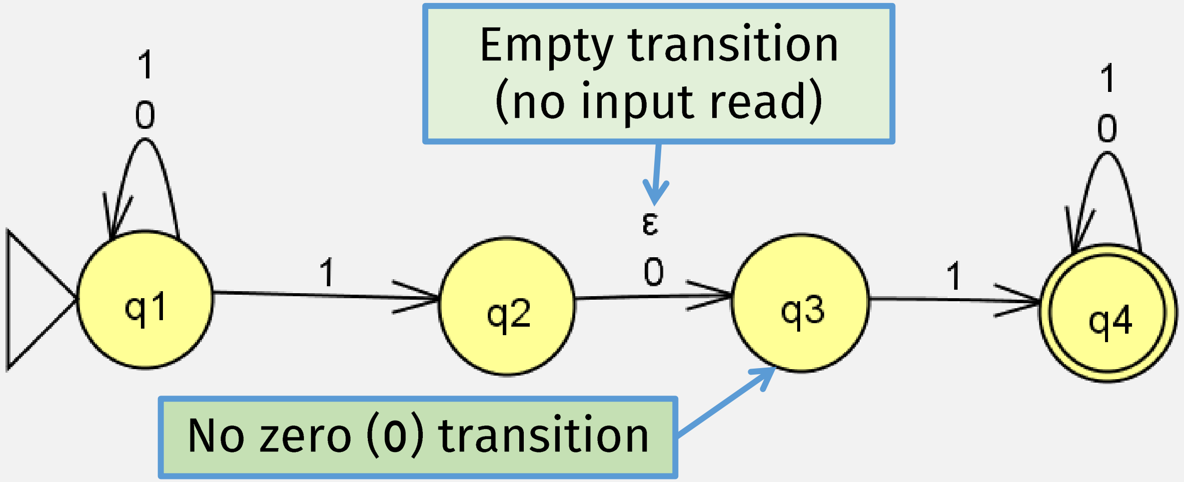
1. $Q = \{q_1, q_2, q_3, q_4\}$,
2. $\Sigma = \{0,1\}$,
3. δ is given as

Empty transition (no input read)

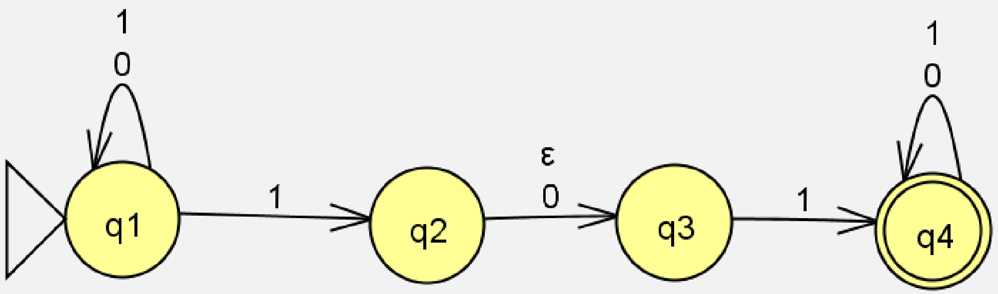
	0	1	ε
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$,

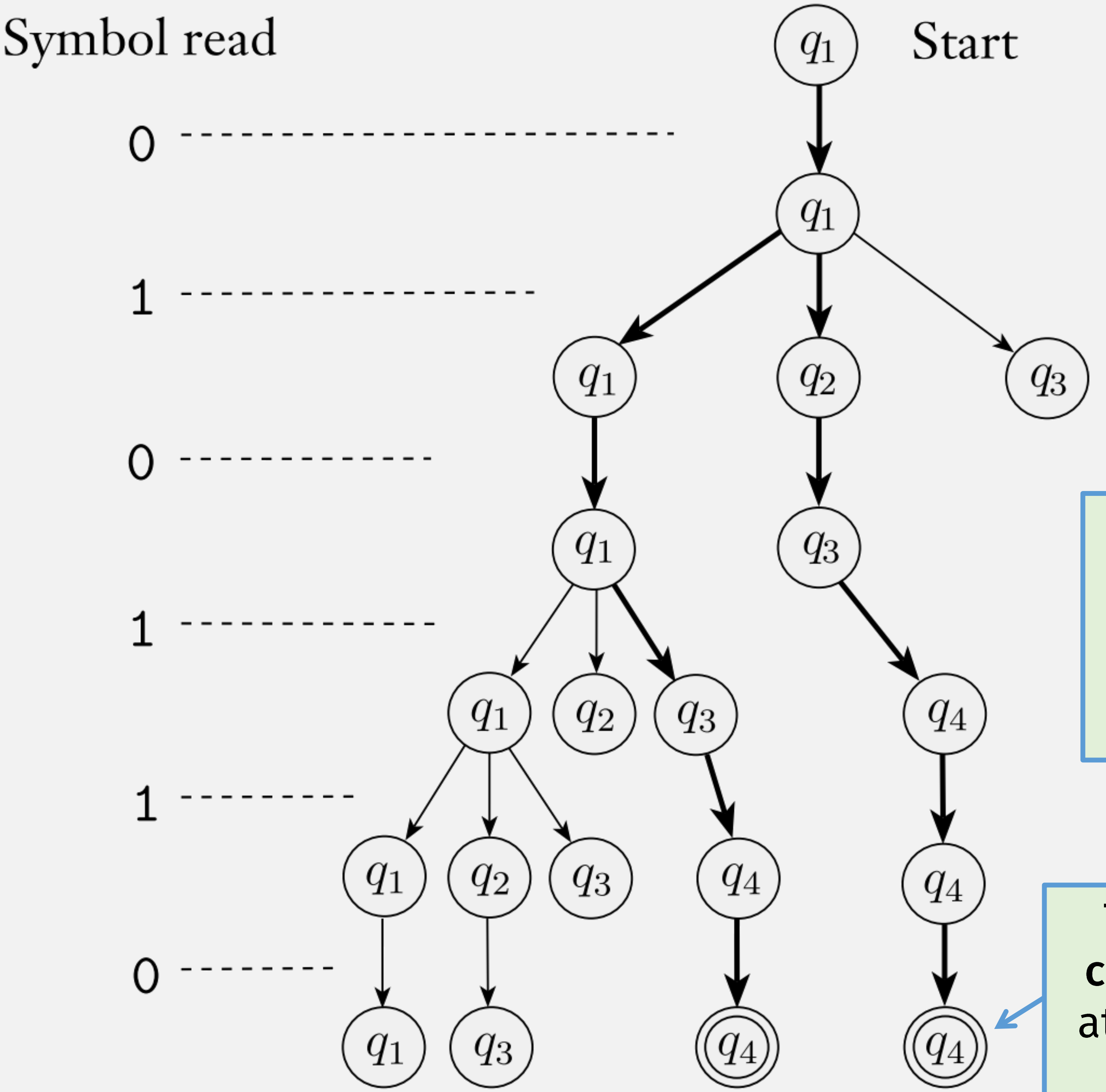
Result of transition can be empty set

4. q_1 is the start state, and
5. $F = \{q_4\}$.



Running Programs, NFAs (JFLAP demo): `010110`





A nondeterministic machine can be in multiple states at the same time!

This is an **accepting computation** because at least one path ends in an accept state

NFAs vs DFAs

DFAs
- Can only be in <u>one</u> state

- Transition:
  - <u>Must read 1 char</u>

- Acceptance:
  - If final state <u>is</u> accept state

NFAs
- Can be in <u>multiple</u> states

- Transition
  - <u>Can read no chars</u>
  - i.e., empty transition

- Acceptance:
  - If <u>one of</u> final states is accept state

No empty transitions

Running an NFA Program: Formal Model

Define the extended transition function: $\hat{\delta} : Q \times \Sigma^* \to \mathcal{P}(Q)$

- *Inputs*:
  - Some beginning state q (not necessarily the start state)
  - Input string $w = w_1 w_2 \cdots w_n$
- *Output*:
  - Set of ending states

(Defined recursively)

- Base case: $\hat{\delta}(q, \epsilon) = \{q\}$
- Recursive case:
  - If: $\hat{\delta}(q, w') = \{q_1, \ldots, q_k\}$ where $w' \in \Sigma^* = w_1 \cdots w_{n-1}$ and $w_n \in \Sigma$
  - Then: $\hat{\delta}(q, w' w_n) = \bigcup_{i=1}^{k} \delta(q_i, w_n)$

All except last char

Combine all possible state transitions for last char

Last char

NFA Extended delta Example

$\hat{\delta}(q, \epsilon) = \{q\}$

$$\hat{\delta}(q, w'w_n) = \bigcup_{i=1}^{k} \delta(q_i, w_n)$$

where $w' \in \Sigma^* = w_1 \cdots w_{n-1}$
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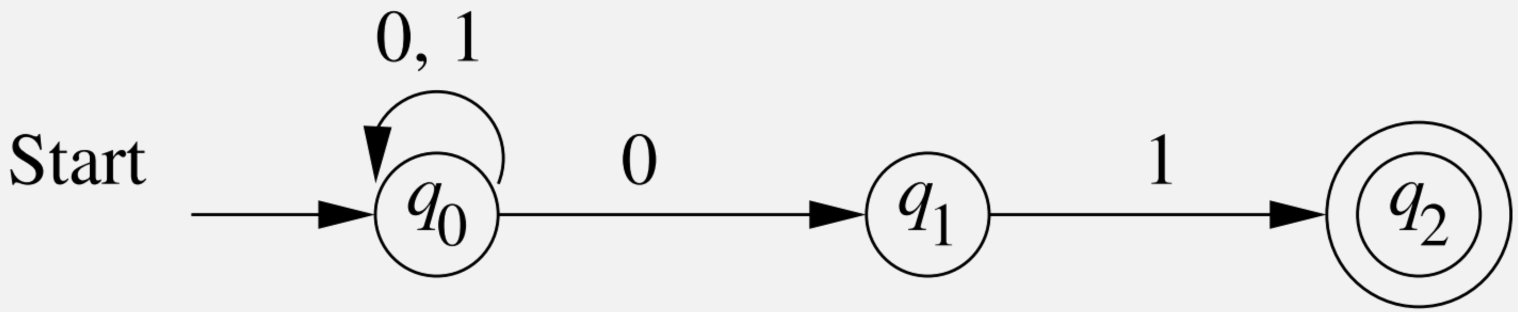
$\hat{\delta}(q, w') = \{q_1, \ldots, q_k\}$



- $\hat{\delta}(q_0, \epsilon) =$
- $\hat{\delta}(q_0, 0) =$
- $\hat{\delta}(q_0, 00) =$
- $\hat{\delta}(q_0, 001) =$

Adding Empty Transitions

- Define the set $\varepsilon\text{-REACHABLE}(q)$
  - … to be all states reachable from q via one or more empty transitions

(Defined recursively)

- Base case: $q \in \varepsilon\text{-REACHABLE}(q)$

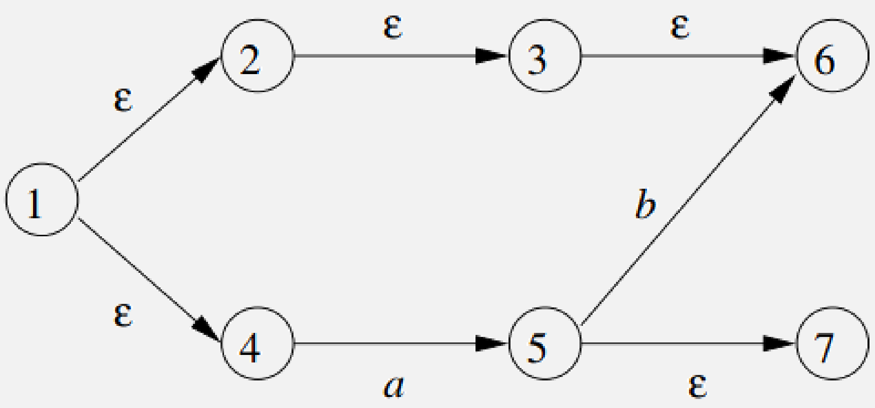
- Inductive case:

$$\varepsilon\text{-REACHABLE}(q) = \{r \mid p \in \varepsilon\text{-REACHABLE}(q) \text{ and } r \in \delta(p, \varepsilon)\}$$

A state is in the reachable set if …

… there is an empty transition to it from another state in the reachable set

ε-REACHABLE Example



$$\varepsilon\text{-REACHABLE}(1) = \{1, 2, 3, 4, 6\}$$

With empty transitions

Running an NFA Program: Formal Model

Define the extended transition function: $\hat{\delta} : Q \times \Sigma^* \to \mathcal{P}(Q)$

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  - Then: $\hat{\delta}(q, w'w_n) = \varepsilon\text{-REACHABLE}(\bigcup_{i=1}^{k} \delta(q_i, w_n))$

s transitions for last char

An NFA's Language

Reminder:
DFA M ***accepts*** w if $\hat{\delta}(q_0, w)$ is in F

- For NFA $N = (Q, \Sigma, \delta, q_0, F)$ N ***accepts*** w if $\hat{\delta}(q_0, w) \cap F \neq \emptyset$
  - i.e., if the final states have at least one accept state
- Language of N = $L(M)$ = $\{w \mid \hat{\delta}(q_0, w) \cap F \neq \emptyset\}$
- Q: How does an NFA's language relate to regular languages
  - Reminder: A language is regular if a **DFA** recognizes it

NFAs and Regular Languages

Theorem:

- A language A is regular **if and only if** some NFA N recognizes it.

How to Prove a Theorem: $X \Leftrightarrow Y$

- $X \Leftrightarrow Y$ = "X if and only if Y" = X iff Y = X <=> Y
- Proof <u>at minimum</u> has 2 parts:

1. **=>** if X, then Y
   - i.e., assume X, then use it to prove Y
   - "forward" direction
2. **<=** if Y, then X
   - i.e., assume Y, then use it to prove X
   - "reverse" direction

NFAs and Regular Languages

<u>*Theorem*</u>:

- A language A is regular **if and only if** some NFA N recognizes it.

- Must prove:
  - **=>** If A is regular, then some NFA N recognizes it
    - Easier
    - <u>We know</u>: if A is regular, then a **DFA** recognizes it.
    - Easy to convert DFA to an NFA! (see HW2)
  - **<=** If an NFA N recognizes A, then A is regular.
    - Harder
    - <u>Idea</u>: Convert NFA to DFA

How to convert **NFA→DFA**?

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

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A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

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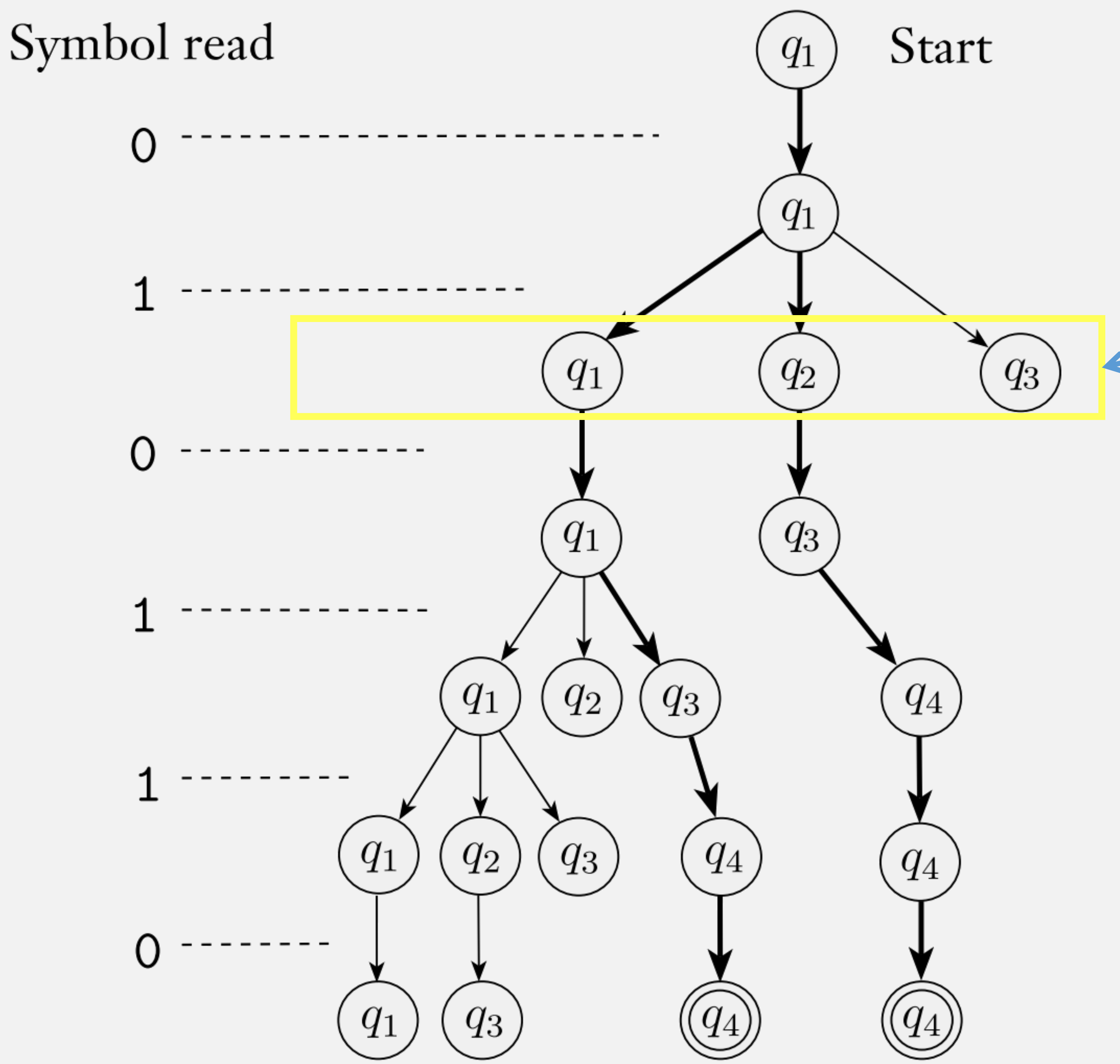
Proof idea:
Let each "state" of the DFA be a set of states in the NFA



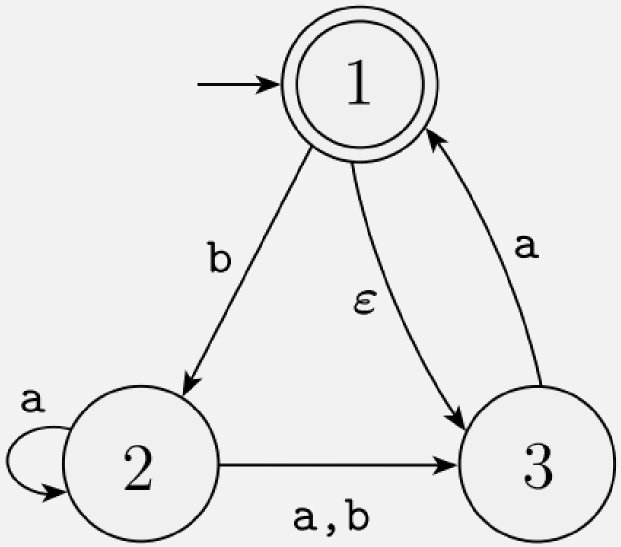
In a DFA, all these states at each step must be only **one** state

So design a state in the DFA to be a **set of NFA states**!

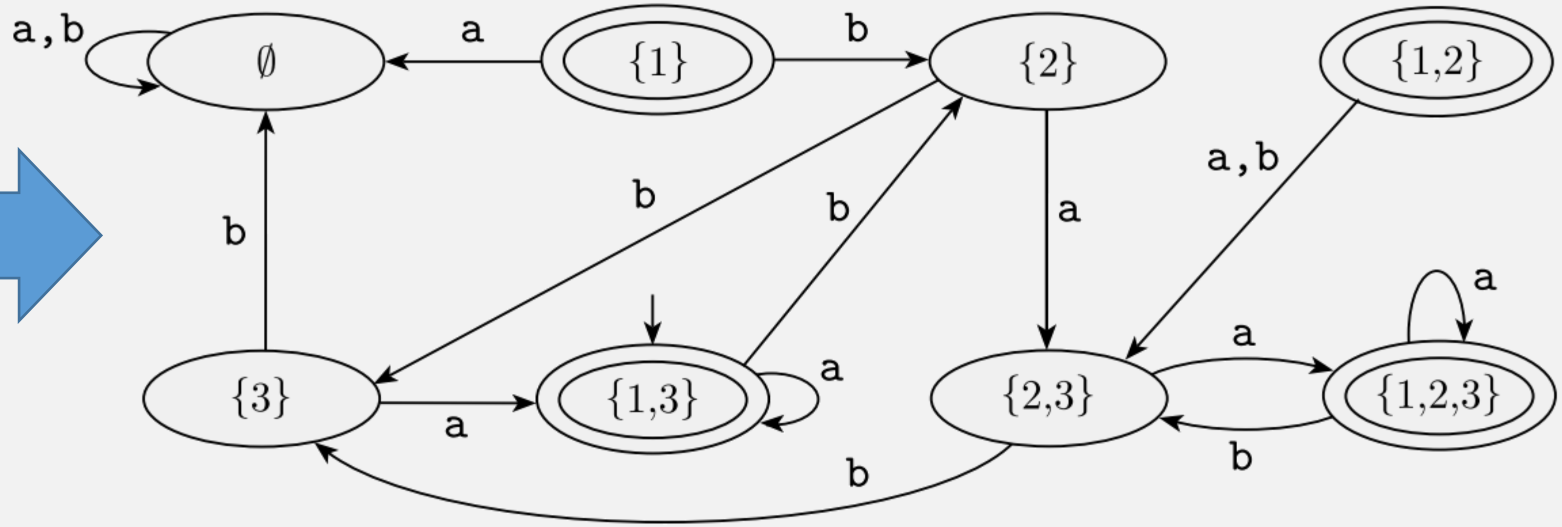
Convert **NFA→DFA**, Formally

- Let NFA N = $(Q, \Sigma, \delta, q_0, F)$
- An equivalent DFA M has states Q' = $\mathcal{P}(Q)$ (power set of Q)

Example:



The NFA N_4



A DFA D that is equivalent to the NFA N_4

No empty transitions

NFA→DFA

Is this correct?

Have: $N = (Q, \Sigma, \delta, q_0, F)$

Want to: construct a DFA $M = (Q', \Sigma, \delta', q_0', F')$

1. $Q' = \mathcal{P}(Q)$.

A state for *M* is a set of states in *N*

2. For $R \in Q'$ and $a \in \Sigma$,

R = a state in *M* = a set of states in *N*

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a)$$

To compute next state for *R*,
compute next states of each NFA state *r* in *R*,
then union results into one set

3. $q_0' = \{q_0\}$

4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$

No empty transitions

NFA→DFA Proof of Correctness

- Let $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$
- And let NFA→DFA(N) = $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$

- Correctness criteria: $L(N) = L(D)$

- We will prove a stronger statement: $\hat{\delta}_D(\{q_0\}, w) = \hat{\delta}_N(q_0, w)$
  - That is, for all strings w, the DFA and NFA end in the same set of states

No empty transitions

NFA→DFA Proof of Correctness

- Let $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$
- And let NFA→DFA(N) = $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$

This produces a set bc we defined states to be sets of states

Theorem: $\hat{\delta}_D(\{q_0\}, w) = \hat{\delta}_N(q_0, w)$

This produces a set bc of the definition of NFAs

Proof: (by induction on length of w)

- Base case $w = \epsilon$ $\hat{\delta}_D(\{q_0\}, \epsilon)$ and $\hat{\delta}_N(q_0, \epsilon)$ are $\{q_0\}$
- Inductive case $w = xa$
  - IH: $\hat{\delta}_D(\{q_0\}, x) = \hat{\delta}_N(q_0, x)$, call this set of states R
  - NFA last step (from δ_N definition) $\bigcup_{r \in R} \delta_N(r, a)$
  - DFA last step (from NFA→DFA definition) $\bigcup_{r \in R} \delta_N(r, a)$

Go back and review previous definitions to confirm

With empty transitions

NFA→DFA$_\varepsilon$

- Have: $N = (Q, \Sigma, \delta, q_0, F)$
- Want to: construct a DFA $M = (Q', \Sigma, \delta', q_0', F')$

1. $Q' = \mathcal{P}(Q)$.

Almost the same, except ...

2. For $R \in Q'$ and $a \in \Sigma$,

$$\delta'(R, a) = \bigcup_{r \in R} \sout{\delta(r, a)}\ \varepsilon\text{-REACHABLE}(\delta(r, a))$$

3. $q_0' = \sout{\{q_0\}}\ \varepsilon\text{-REACHABLE}(\{q_0\})$

4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$

With empty transitions

NFA→DFA$_\varepsilon$ Proof of Correctness

- Let $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$
- And let NFA→DFA$_\varepsilon$(N) =$D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$

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(Same as before)

With empty transitions

NFA→DFA$_\varepsilon$ Proof of Correctness

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Almost the same, except …

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  - NFA last step (from δ_N definition) $\bigcup_{r \in R} \delta_N(r, a)$
  - DFA last step (from NFA→DFA definition) $\bigcup_{r \in R} \delta_N(r, a)$

Proving that NFAs Recognize Reg Langs

Theorem:

A language A is regular **if and only if** some NFA N recognizes it.

Proof:

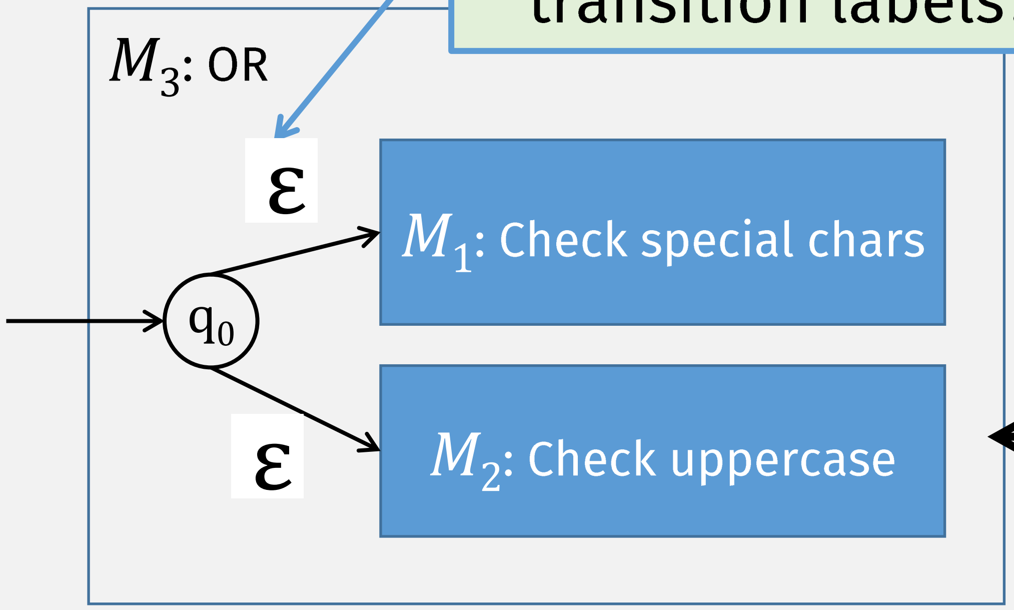
=> If A is regular, then some NFA N recognizes it

- We know: if A is regular, then a DFA recognizes it
- So convert DFA to an NFA

<= If an NFA N recognizes A, then A is regular

- We know: if a DFA recognizes a language, then it is regular
- So convert NFA to DFA …
- … Using NFA→DFA algorithm we just defined! ∎ (Q.E.D.)

Combining DFAs



Problem 1: ☑
What should be the transition labels?

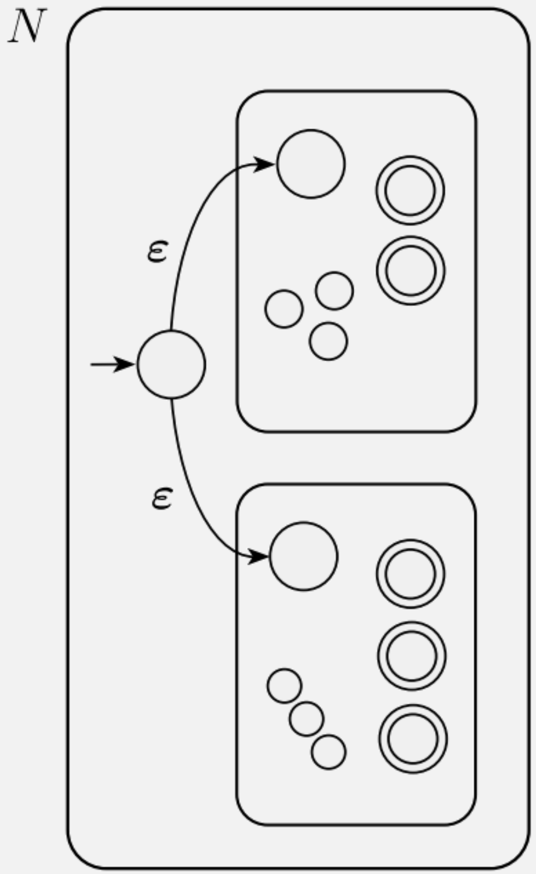
Problem 2: ☑
Once we enter one of the machines, can't go back to the other one!

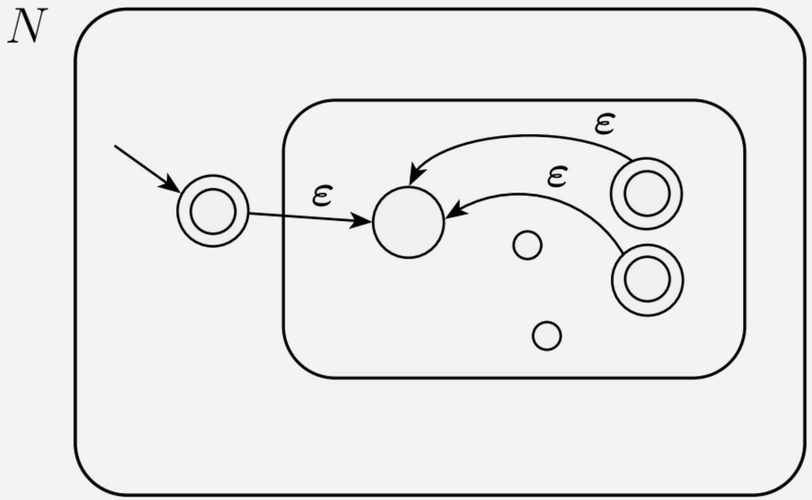
This is an **NFA**! Can be in multiple states at once, but is still equivalent to a regular language!

…ent …e!

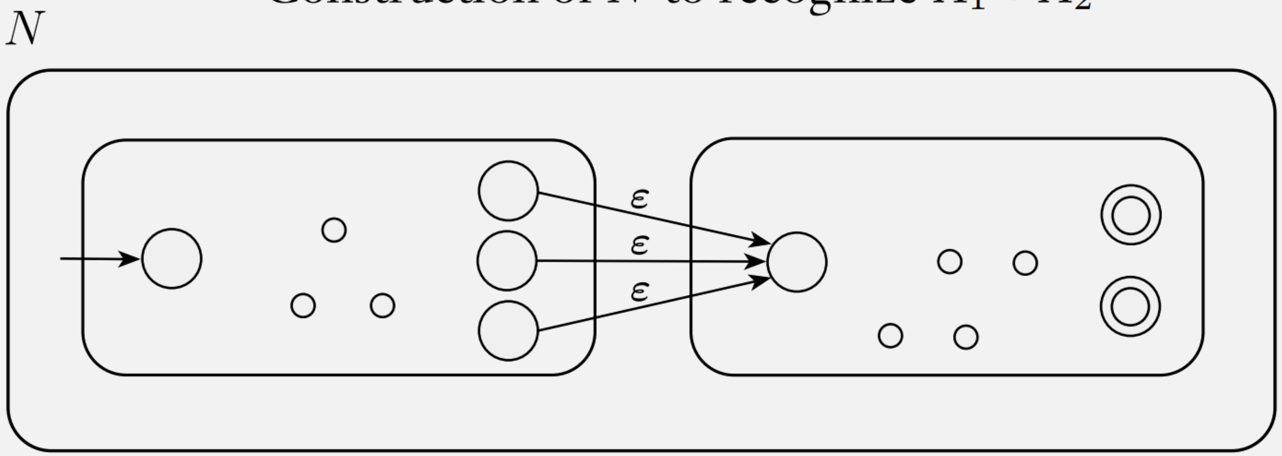
I… allows b… s (i.e., … at once!

Next Time: More "Combining" Operations





Construction of N to recognize $A_1 \circ A_2$



In-class Quiz 9/15

On gradescope