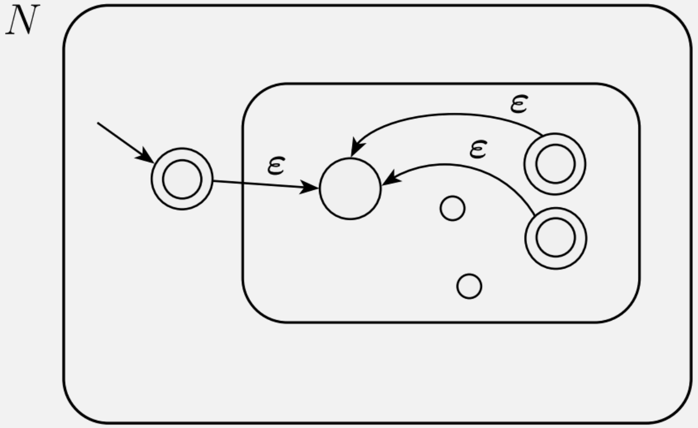
UMB CS622

Closed Operations on Regular Languages

Monday, September 20, 2021



Announcements

- HW1 due yesterday
- HW2 released, due Sun 9/26 11:59pm EST
- Reminder: Post HW questions to Piazza
  - Use anonymous post if you don’t want anyone to see
- Midterm / Final exam cancelled

Last Time: NFAs vs DFAs

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta : Q \times \Sigma \longrightarrow Q$ is the ***transition function***,
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

DFAs

- Can only be in one state
- Transitions:
  - Always reads one char
  - A state must have a transition for every char
- Acceptance:
  - If final state is accept state

NFAs

- Can be in multiple states
- Transitions:
  - Can read no chars, i.e., empty transition
  - A state might not have transitions for every char
- Acceptance:
  - If one of final states is accept state

Last Time: NFAs and Regular Languages

Theorem:

A language A is regular **if and only if** some NFA N recognizes it.

Proof:

=> If A is regular, then some NFA N recognizes it

- Easier
- We know: if A is regular, then a **DFA** recognizes it.
- Convert DFA to an NFA! (see HW2)

<= If an NFA N recognizes A, then A is regular.

- Harder
- We know: a language is regular if a **DFA** recognizes it.
- Convert NFA to DFA

Last Time: How to convert **NFA→DFA**?

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
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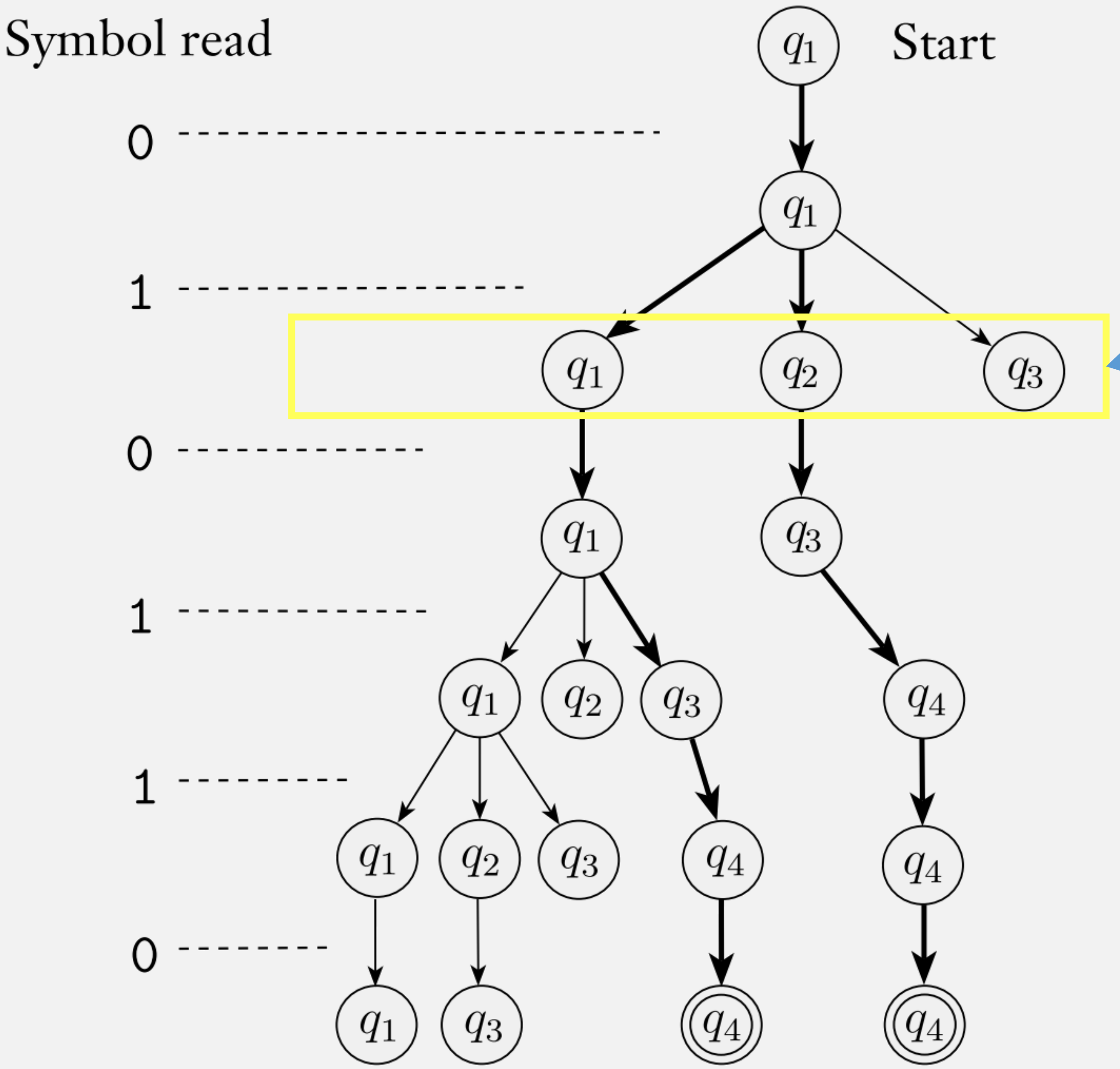
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Proof idea:
Let each "state" of the DFA be a set of states in the NFA



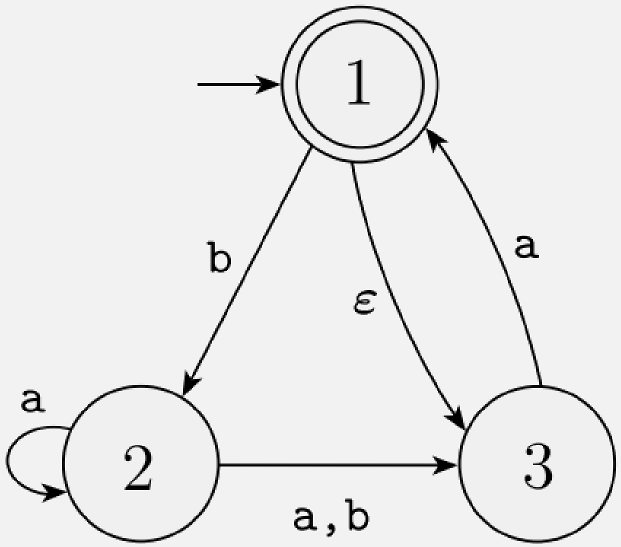
In a DFA, all these states at each step must be only **one** state

So design a state in the DFA to be a **set of NFA states**!

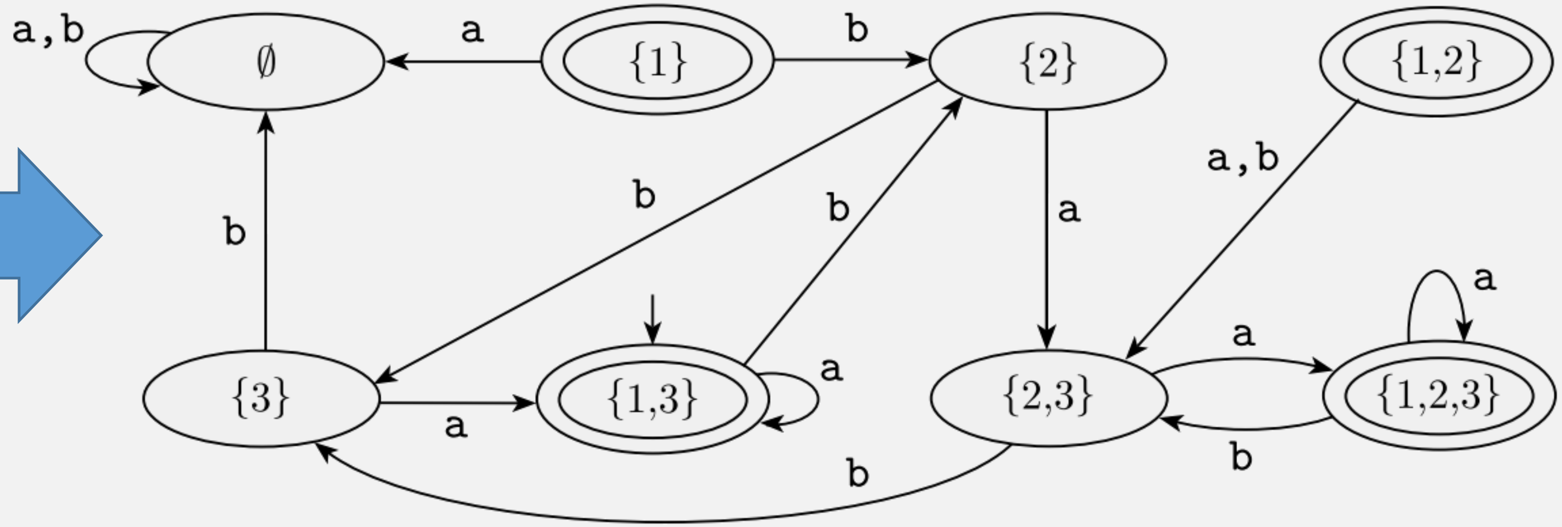
Convert **NFA→DFA**, Formally

- Let NFA N = $(Q, \Sigma, \delta, q_0, F)$
- An equivalent DFA M has states Q' = $\mathcal{P}(Q)$ (power set of Q)

Example:



The NFA N_4



A DFA D that is equivalent to the NFA N_4

No empty transitions

NFA→DFA

Is this correct?

Have: $N = (Q, \Sigma, \delta, q_0, F)$

Want to: construct a DFA $M = (Q', \Sigma, \delta', q_0', F')$

1. $Q' = \mathcal{P}(Q)$

A state for *M* is a set of states in *N*

2. For $R \in Q'$ and $a \in \Sigma$,

$$\delta'(R, a) = \bigcup_{r \in R} \delta(r, a)$$

To compute a single step in the DFA … compute next states of each NFA state *r* in *R*, then union results together

3. $q_0' = \{q_0\}$

R = a state in *M* = a set of states in *N*

4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$

No empty transitions

NFA→DFA Proof of Correctness

Let $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$

And let **NFA→DFA(N) =** $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$

Correctness criteria: LANGUAGEOF(N) = LANGUAGEOF(D)

- I.e., for all strings w, N accepts w **if and only if** D accepts w

- We will first prove a stronger statement: $\hat{\delta}_D(\{q_0\}, w) = \hat{\delta}_N(q_0, w)$
  - I.e., for all strings w, the DFA and NFA end in the same set of states!

Remember:
A state in the DFA is a set of states in the NFA

No empty transitions

NFA→DFA Proof of Correctness

Let $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$

And let **NFA→DFA(*N*)** = $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$

This produces a set bc we defined states to be sets of states

Theorem: $\hat{\delta}_D(\{q_0\}, w) = \hat{\delta}_N(q_0, w)$

This produces a set bc of the definition of NFAs

Proof: (by induction on length of *w*)

- Base case $w = \epsilon$ $\hat{\delta}_D(\{q_0\}, \epsilon)$ and $\hat{\delta}_N(q_0, \epsilon)$ =
- Inductive case $w = xa$ ← *a* = last char
  - IH: $\hat{\delta}_D(\{q_0\}, x) = \hat{\delta}_N(q_0, x)$, call this set of states R
  - NFA last step (from δ_N definition) $\bigcup_{r \in R} \delta_N(r, a)$
  - DFA last step (from NFA→DFA definition) $\bigcup_{r \in R} \delta_N(r, a)$

Go back and review previous definitions to confirm that they are the same

Last Time: Adding Empty Transitions

Define the set $\varepsilon\text{-REACHABLE}(q)$

... to be all states reachable from q via one or more empty transitions

- Base case: $q \in \varepsilon\text{-REACHABLE}(q)$

- Inductive case:

A state is in the reachable set if ...

$$\varepsilon\text{-REACHABLE}(q) = \{r \mid p \in \varepsilon\text{-REACHABLE}(q) \text{ and } r \in \delta(p, \varepsilon)\}$$

... there is an empty transition to it from another state in the reachable set

With empty transitions

NFA→DFA$_\varepsilon$

Have: $N = (Q, \Sigma, \delta, q_0, F)$

Want to: construct a DFA $M = (Q', \Sigma, \delta', q_0', F')$

1. $Q' = \mathcal{P}(Q)$.

Almost the same, except ...

2. For $R \in Q'$ and $a \in \Sigma$,

$$\delta'(R, a) = \bigcup_{r \in R} \sout{\delta(r, a)}\ \varepsilon\text{-REACHABLE}(\delta(r, a))$$

3. $q_0' = \sout{\{q_0\}}\ \varepsilon\text{-REACHABLE}(\{q_0\})$

4. $F' = \{R \in Q' \mid R \text{ contains an accept state of } N\}$

With empty transitions

NFA→DFA$_\varepsilon$ Proof of Correctness

Let $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$

And let **NFA→DFA(N) =** $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$

Correctness criteria: LANGUAGEOF(N) = LANGUAGEOF(D)

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- We will first prove a stronger statement: $\hat{\delta}_D(\{q_0\}, w) = \hat{\delta}_N(q_0, w)$
  - I.e., for all strings w, the DFA and NFA end in the same set of states!

(Same as before)

With empty transitions

NFA→DFA$_\varepsilon$ Proof of Correctness

Let $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$

And let **NFA→DFA(N) =** $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$

Theorem: $\hat{\delta}_D(\{q_0\}, w) = \hat{\delta}_N(q_0, w)$

Almost the same, except ...

Proof: (by induction on length of w)

- Base case $w = \epsilon$ $\hat{\delta}_D(\{q_0\}, \epsilon)$ and $\hat{\delta}_N(q_0, \epsilon)$ =
- Inductive case $w = xa$ ← a = last char
  - IH: $\hat{\delta}_D(\{q_0\}, x) = \hat{\delta}_N(q_0, x)$, call this set of states R ?????
  - NFA last step (from δ_N definition) $\bigcup_{r \in R} \delta_N(r, a)$
  - DFA last step (from NFA→DFA definition) $\bigcup_{r \in R} \delta_N(r, a)$

NFA→DFA Proof of Correctness

Let $N = (Q_N, \Sigma, \delta_N, q_0, F_N)$

And let **NFA→DFA**(N) = $D = (Q_D, \Sigma, \delta_D, \{q_0\}, F_D)$

Last Step ...
(see hw2)

Correctness criteria: LANGUAGEOF(N) = LANGUAGEOF(D)

- I.e., for all strings w, N accepts w **if and only if** D accepts w

- We will first prove a stronger statement: $\hat{\delta}_D(\{q_0\}, w) = \hat{\delta}_N(q_0, w)$
  - I.e., for all strings w, the DFA and NFA end in the same set of states!

Proving that NFAs Recognize Reg Langs

Theorem:

A language A is regular **if and only if** some NFA N recognizes it.

I.e., NFAs also represent regular languages!

Proof:

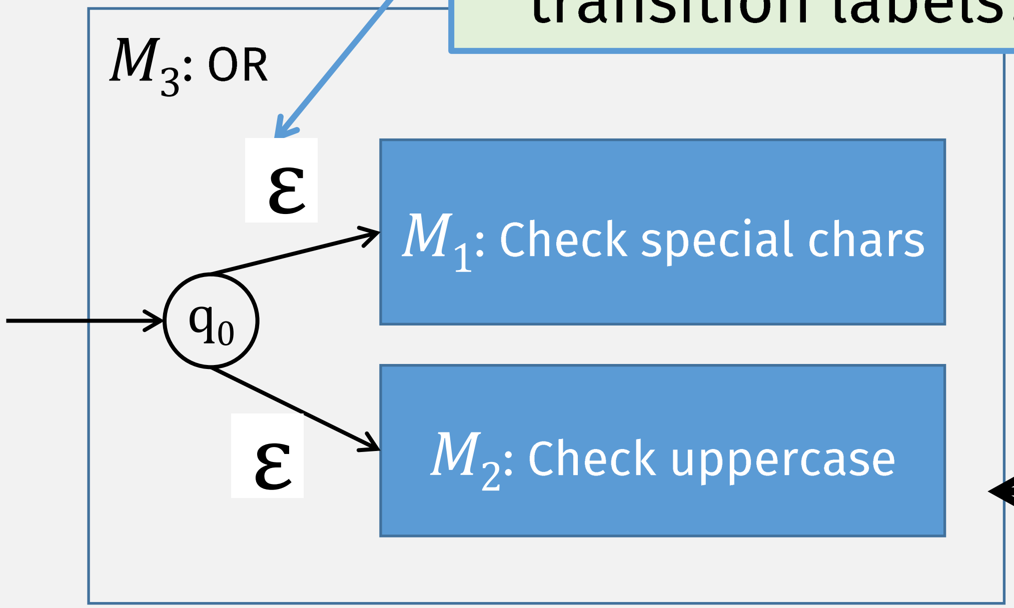
=> If A is regular, then some NFA N recognizes it

- We know: If A is regular, then a DFA recognizes it
- So convert that DFA to an NFA

<= If an NFA N recognizes A, then A is regular

- We know: A language is regular if there is a DFA recognizing it
- So convert NFA to DFA …
- … Using NFA→DFA algorithm we just defined! ■ (Q.E.D.)

Last Time: Combining DFAs



Problem 1: What should be the transition labels?

Problem 2: Once we enter one of the machines, can't go back to the other one!

We need a different

This is an **NFA!**
Can be in multiple states, and is still equivalent to a regular language!

tiple machines) at once!

Why is this so important!?

Combine machines again!

Combine machines

Password Requirements

- Passwords must have a minimum length of ten (10) characters - but more is better! ← DFA
- Passwords **must include at least 3** different types of characters:
  - upper-case letters (A-Z) ← DFA
  - lower-case letters (a-z) ← DFA
  - symbols or special characters (%, &, *, $, etc.) ← DFA
  - numbers (0-9) ← DFA
- Passwords cannot contain all or part of your email address ← DFA
- Passwords cannot be re-used ← DFA

<u>Review</u>: "Closed" Operations

- Natural numbers = {0, 1, 2, …}
  - <u>Closed</u> under addition: if x and y are Natural, then z = x + y is a Nat
  - Closed under multiplication?
    - yes
  - Closed under subtraction?
    - no
- Integers = {…, -2, -1, 0, 1, 2, …}
  - Closed under addition and multiplication
  - Closed under subtraction?
    - yes
  - Closed under division?
    - no
- Rational numbers = {x | x = y/z, y and z are ints}
  - Closed under division?
    - No?
    - Yes if z !=0

A set is <u>**closed**</u> under an operation if … applying it to members of the set returns a member in the set

Why Care About Closed Operations?

- Because it allows repeatedly applying an operation to a set
- E.g., Closed operations on regular languages preserves “regularness”
- So result of combining DFAs/NFAs can be combined again and again

Operations on Regular Languages

Let A and B be languages. We define the regular operations ***union***, ***concatenation***, and ***star*** as follows:

- **Union**: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
- **Concatenation**: $A \circ B = \{xy \mid x \in A \text{ and } y \in B\}$.
- **Star**: $A^* = \{x_1 x_2 \dots x_k \mid k \geq 0 \text{ and each } x_i \in A\}$.

Union: $A \cup B = \{x | \ x \in A \text{ or } x \in B\}$

Union Example

Let the alphabet Σ be the standard 26 letters $\{\texttt{a}, \texttt{b}, \ldots, \texttt{z}\}$.

If $A = \{\texttt{good}, \texttt{bad}\}$ and $B = \{\texttt{boy}, \texttt{girl}\}$, then

$$A \cup B = \{\texttt{good}, \texttt{bad}, \texttt{boy}, \texttt{girl}\}$$

Union: $A \cup B = \{x | x \in A \text{ or } x \in B\}$

Union is Closed for Regular Languages

THEOREM

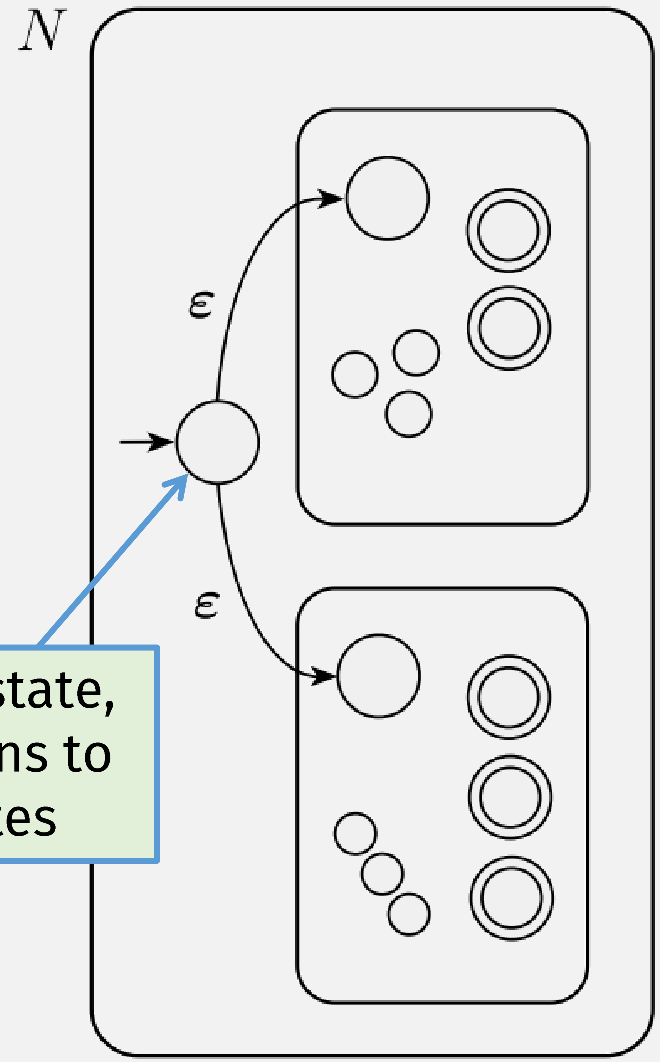
The class of regular languages is closed under the union operation.

In other words, if A_1 and A_2 are regular languages, so is $A_1 \cup A_2$.

Proof:

- How do we prove that a language is regular?
  - Create a DFA/NFA recognizing it!
- Create machine combining the machines recognizing A_1 and A_2
  - Should we create a DFA or NFA?

Union is Closed for Regular Languages



Add new start state, and ε-transitions to old start states

Union is Closed for Regular Languages

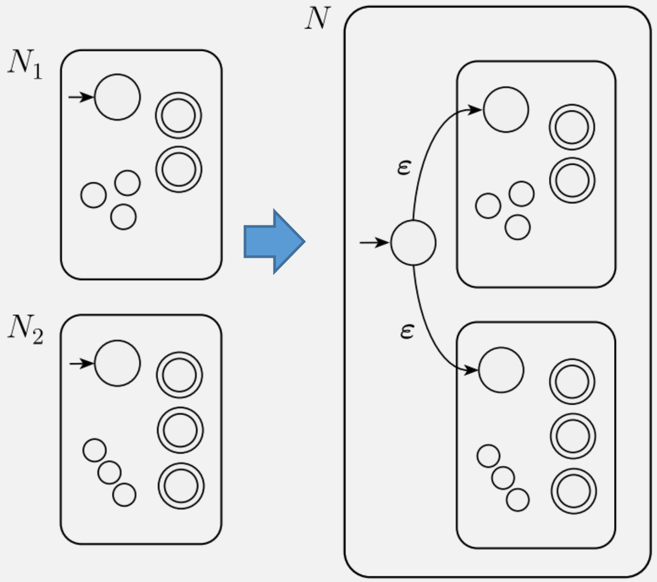
PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize $A_1 \cup A_2$.

1. $Q = \{q_0\} \cup Q_1 \cup Q_2$.
2. The state q_0 is the start state of N.
3. The set of accept states $F = F_1 \cup F_2$.
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \\ \delta_2(q,a) & q \in Q_2 \\ \{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon \end{cases}$$



Union is Closed for Regular Languages

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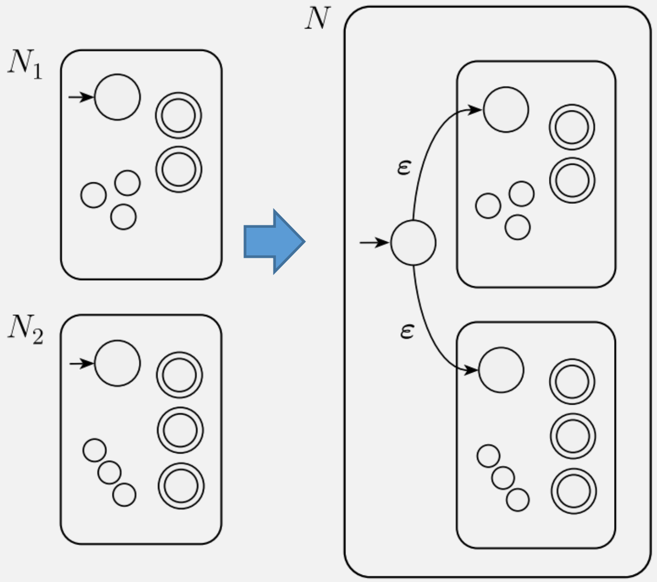
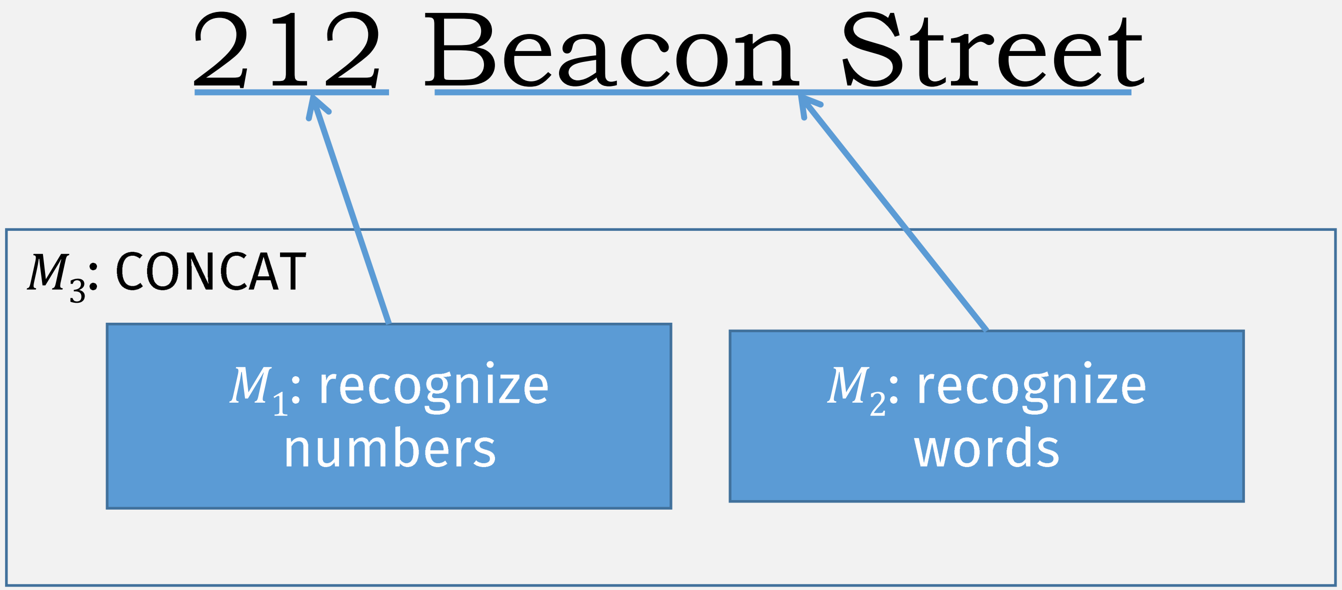


Another operation: Concatenation

- Example: Matching street addresses



Concatenation: $A \circ B = \{xy | \, x \in A \text{ and } y \in B\}$

Concatenation Example

Let the alphabet Σ be the standard 26 letters $\{\texttt{a}, \texttt{b}, \ldots, \texttt{z}\}$.

If $A = \{\texttt{good}, \texttt{bad}\}$ and $B = \{\texttt{boy}, \texttt{girl}\}$, then

$$A \circ B = \{\texttt{goodboy}, \texttt{goodgirl}, \texttt{badboy}, \texttt{badgirl}\}$$

Concatenation: $A \circ B = \{xy | x \in A \text{ and } y \in B\}$

Concatenation is Closed

THEOREM

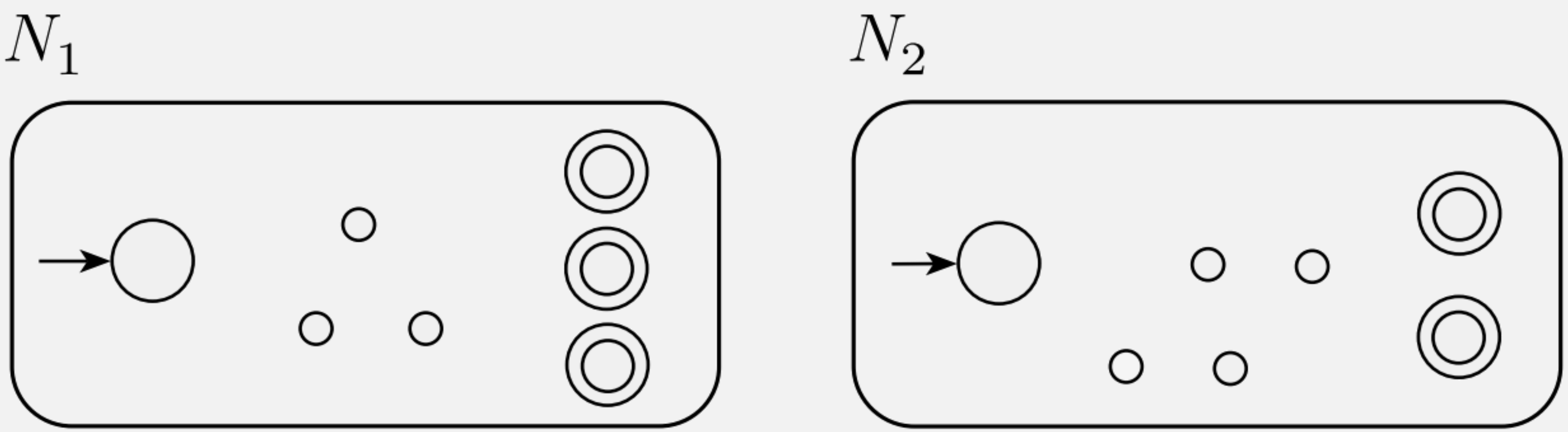
The class of regular languages is closed under the concatenation operation.

In other words, if A_1 and A_2 are regular languages then so is $A_1 \circ A_2$.

Proof: Construct a new machine? (like union)

- How does it know when to switch from N_1 to N_2?
  - Can only read input once

Concatentation



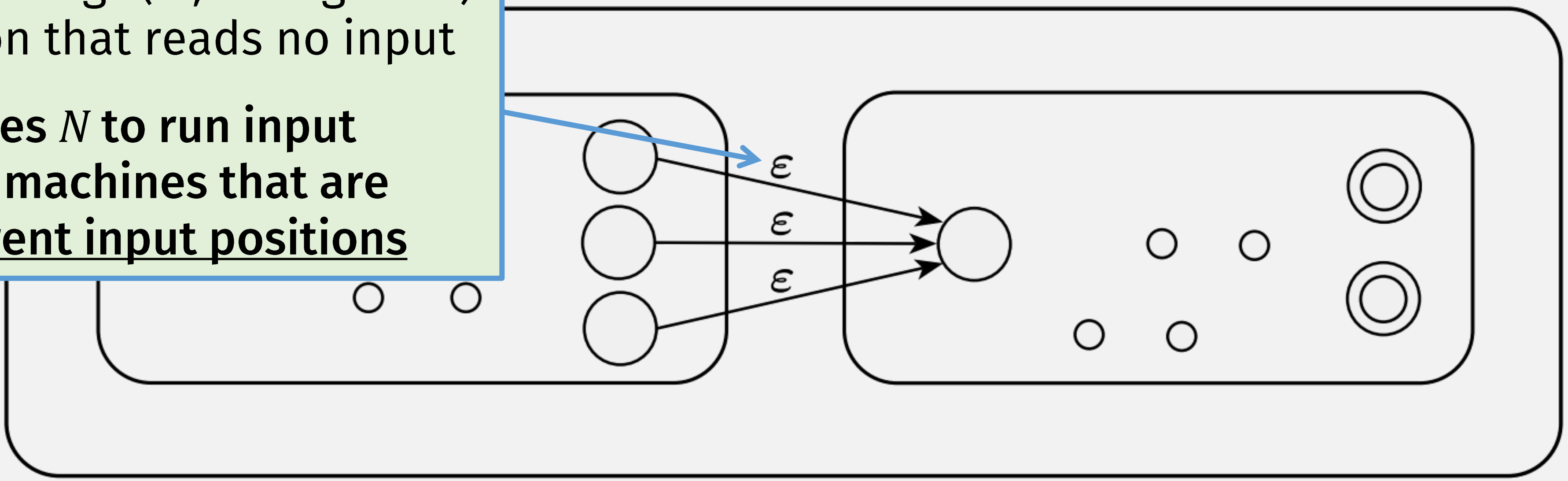
Let N_1 recognize A_1, and N_2 recognize A_2.

N must <u>simultaneously</u>:
- Keep checking with N_1 **and**
- Move to N_2 to check 2nd part

<u>Want</u>: Construction of N to recognize $A_1 \circ A_2$

ε = "empty string" (ie, 0 length str)
= transition that reads no input

Enables N to run input on two machines that are <u>at different input positions</u>



Concatenation is Closed for Regular Langs

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
$N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognize A_2.

Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

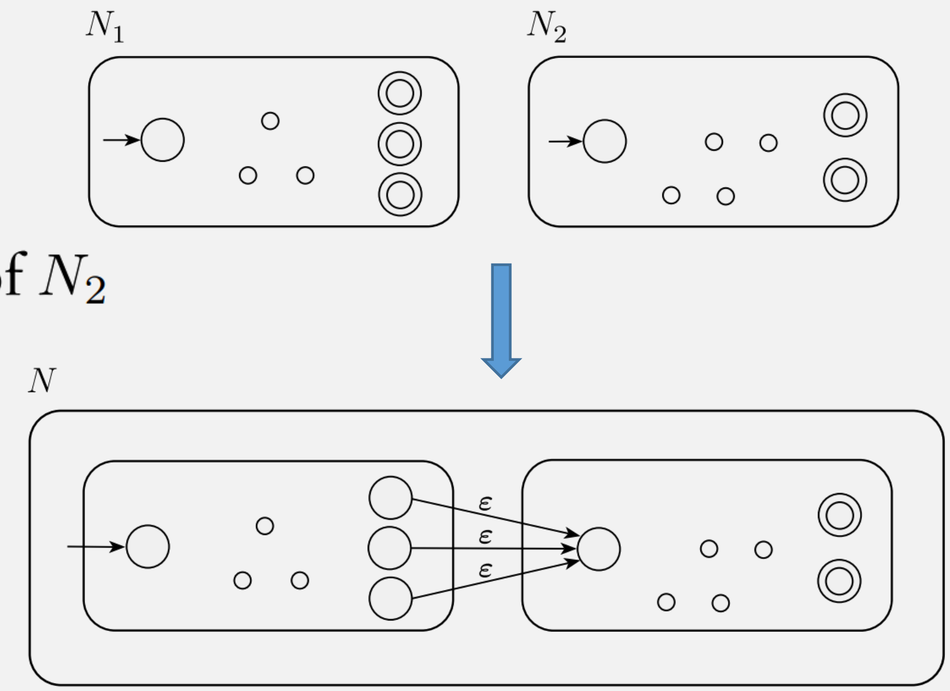
1. $Q = Q_1 \cup Q_2$

2. The state q_1 is the same as the start state of N_1

3. The accept states F_2 are the same as the accept states of N_2

4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \boldsymbol{\varepsilon} \\ \delta_1(q,a) \cup \{q_2\} & q \in F_1 \text{ and } a = \boldsymbol{\varepsilon} \\ \delta_2(q,a) & q \in Q_2. \end{cases}$$



Concatenation is Closed for Regular Langs

PROOF

Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1, and
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Construct $N = (Q, \Sigma, \delta, q_1, F_2)$ to recognize $A_1 \circ A_2$

1. $Q = Q_1 \cup Q_2$

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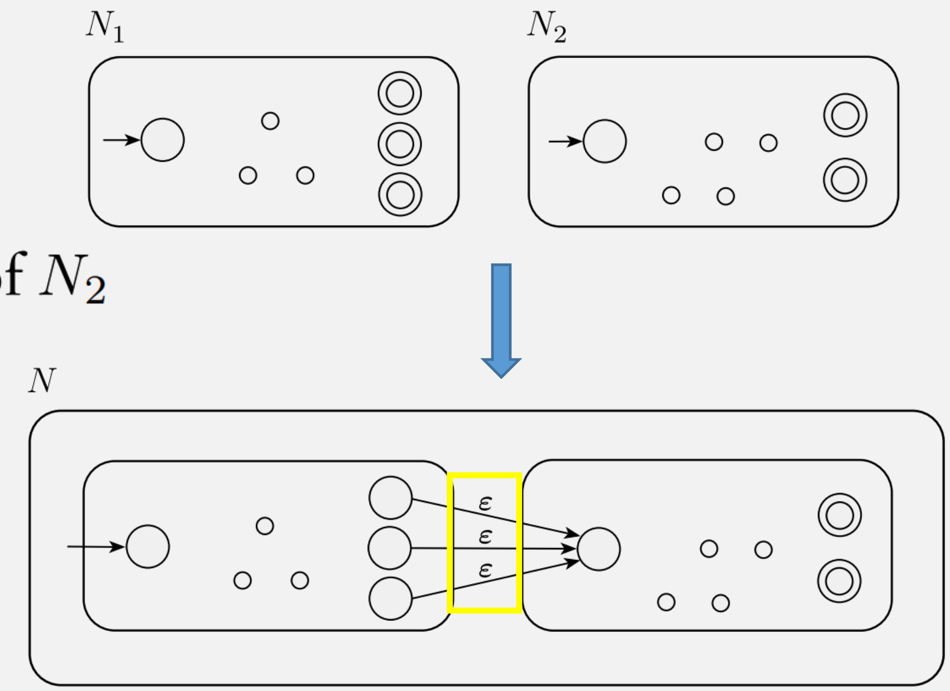
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q, a) = \begin{cases} ? \\ ? \\ ? \\ ? \end{cases}$$



Star: $A^* = \{x_1x_2\ldots x_k|\ k \geq 0 \text{ and each } x_i \in A\}$

(Kleene) Star Example

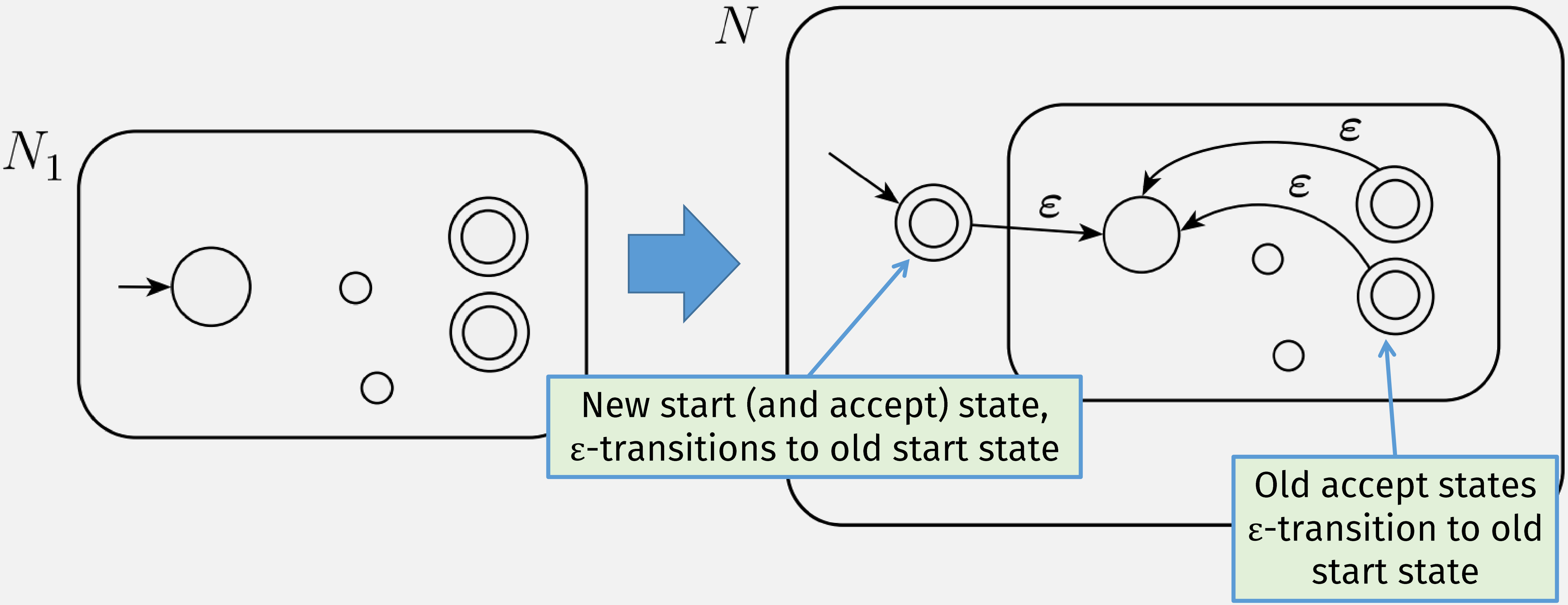
Let the alphabet Σ be the standard 26 letters $\{\mathtt{a}, \mathtt{b}, \ldots, \mathtt{z}\}$.

If $A = \{\mathtt{good}, \mathtt{bad}\}$ and $B = \{\mathtt{boy}, \mathtt{girl}\}$, then

$$A^* = \begin{array}{l}\{\varepsilon, \text{good, bad, goodgood, goodbad, badgood, badbad,} \\ \text{goodgoodgood, goodgoodbad, goodbadgood, goodbadbad, } \ldots\}\end{array}$$

(this is an infinite language)

Kleene Star

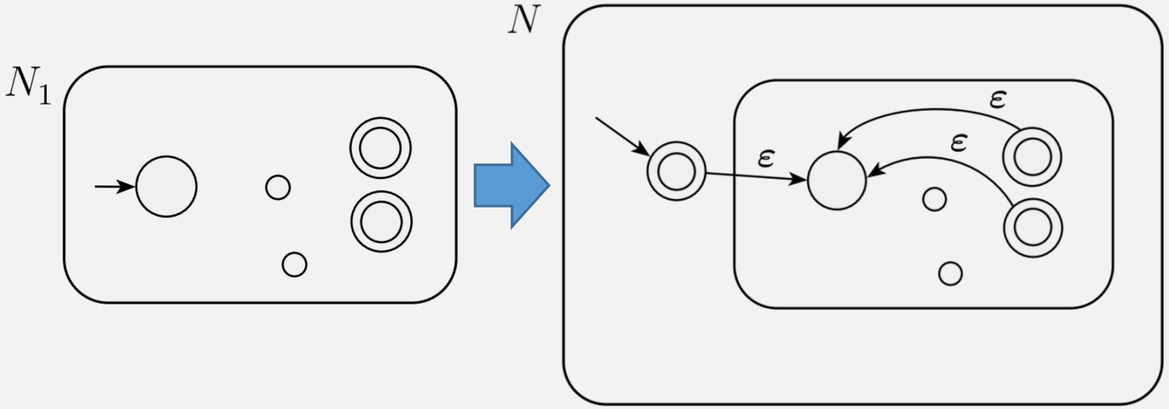

N_1
N
ε
ε
ε
New start (and accept) state,
ε-transitions to old start state
Old accept states
ε-transition to old
start state

Kleene Star is Closed for Regular Langs

THEOREM

The class of regular languages is closed under the star operation.

Kleene Star is Closed for Regular Langs



PROOF Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1. Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize A_1^*.

1. $Q = \{q_0\} \cup Q_1$

2. The state q_0 is the new start state.

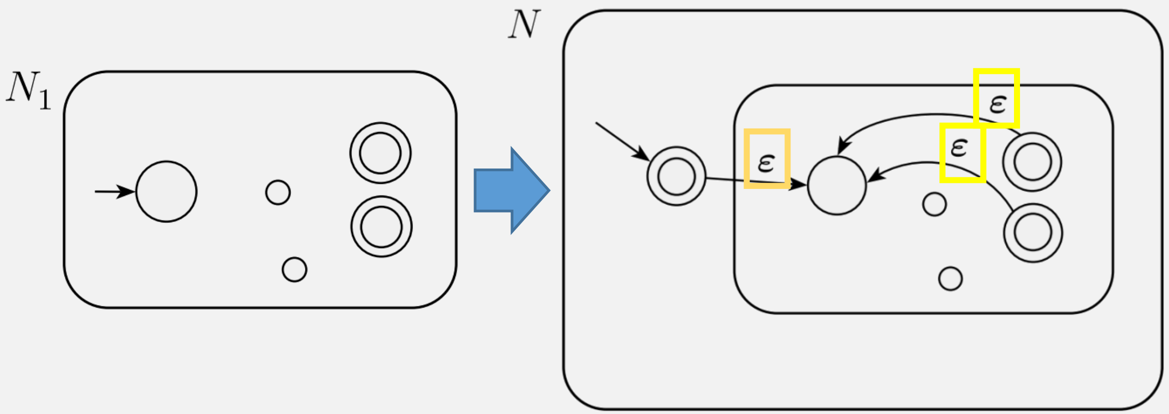
3. $F = \{q_0\} \cup F_1$

4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

$$\delta(q,a) = \begin{cases} \delta_1(q,a) & q \in Q_1 \text{ and } q \notin F_1 \\ \delta_1(q,a) & q \in F_1 \text{ and } a \neq \varepsilon \\ \delta_1(q,a) \cup \{q_1\} & q \in F_1 \text{ and } a = \varepsilon \\ \{q_1\} & q = q_0 \text{ and } a = \varepsilon \\ \emptyset & q = q_0 \text{ and } a \neq \varepsilon. \end{cases}$$

Kleene Star is Closed for Regular Langs

PROOF Let $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognize A_1. Construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize A_1^*.



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2. The state q_0 is the new start state.
3. $F = \{q_0\} \cup F_1$
4. Define δ so that for any $q \in Q$ and any $a \in \Sigma_\varepsilon$,

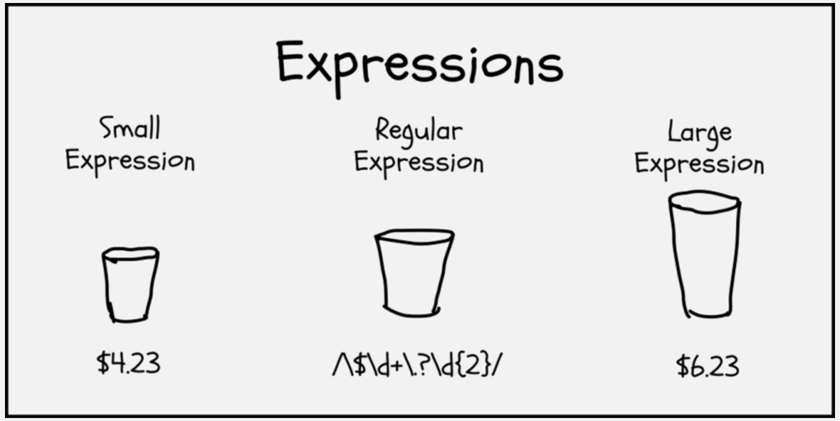
$$\delta(q, a) = \begin{cases} ? \\ ? \\ ? \\ ? \\ ? \end{cases}$$

Kleene star of a language must accept the empty string!

Many More Closed Operations on Regular Languages!

- Complement
- Intersection
- Difference
- Reversal
- Homomorphism
- (See HW2)

Next Time: Regular Expressions



In-class quiz 9/20

See Gradescope