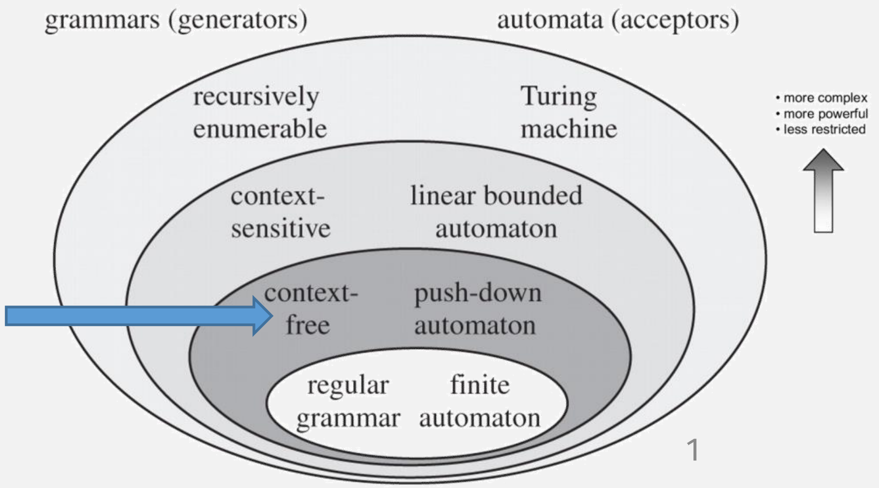
CS622

Context-Free Languages (CFLs)

Wednesday, September 29, 2021



Announcements

- First **in-person class**: next Monday 10/4 7pm
  - McCormack M01-0209
- HW3 due Sun 11:59pm EST
- HW2 grades released

HW2 Review

2 Exending the definition of "REACHABLE"

Define ε- REACHABLE$_{qs}$, which is like the ε- REACHABLE definition from class, but extended to sets of states. (Don't forget to handle the empty set!)

$$\varepsilon\text{-REACHABLE}_{qs}(qs) = \bigcup_{q \in qs} \varepsilon\text{-REACHABLE}(q)$$

HW2 Review

3 DFA->NFA

In class we showed how to convert an NFA into an an equivalent DFA, but not a DFA to NFA. Do this now.

More specifically:

- Come up with a procedure **DFA→NFA** that converts DFAs to equivalent NFAs. In other words, given some DFA $M = (Q, \Sigma, \delta, q_0, F)$ that satisfies the formal definition of DFAs from class, **DFA→NFA** should produce some NFA $N = (Q', \Sigma, \delta', q_0', F')$ that satisfies the formal definition of NFAs and accepts the same language as M.

3. DFA $M = (Q, \Sigma, \delta, q_0, F)$

To produce NFA $N = (Q_N, \Sigma, \delta_N, q_0', F_N)$

1. $Q_N = Q$
2. $\Sigma = \Sigma$
3. $q_0' = q_0$
4. $F_N = F$
5. δ_N is given as $Q' \times \Sigma_\varepsilon \rightarrow P(Q')$ for $R \in Q_N$ and $a \in \Sigma_\varepsilon$

$$\delta_N(R, a) = \{\delta(R, a)\}$$

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the ***states***,
2. Σ is a finite set called the ***alphabet***,
3. $\delta : Q \times \Sigma \longrightarrow Q$ is the ***transition function***,
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

A ***nondeterministic finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set of states,
2. Σ is a finite alphabet,
3. $\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

HW2 Review

3 DFA->NFA

....

- Then prove that your procedure is correct, i.e., that M accepts some string w **if and only if** N accepts w. You'll probably want to use induction on the length of w.

$\Rightarrow$ If M accepts w, then N accepts w

- If M accepts w, then $\hat{\delta}_M(q_0, w) \in F$ ← Criteria for acceptance for DFAs / NFAs
- So N accepts w because $\hat{\delta}_N(q_0, w) = \{\hat{\delta}_M(q_0, w)\}$ thus $\hat{\delta}_N(q_0, w) \cap F_N \neq \emptyset$

$\Leftarrow$ If N accepts w, the M accepts w

- (similar)

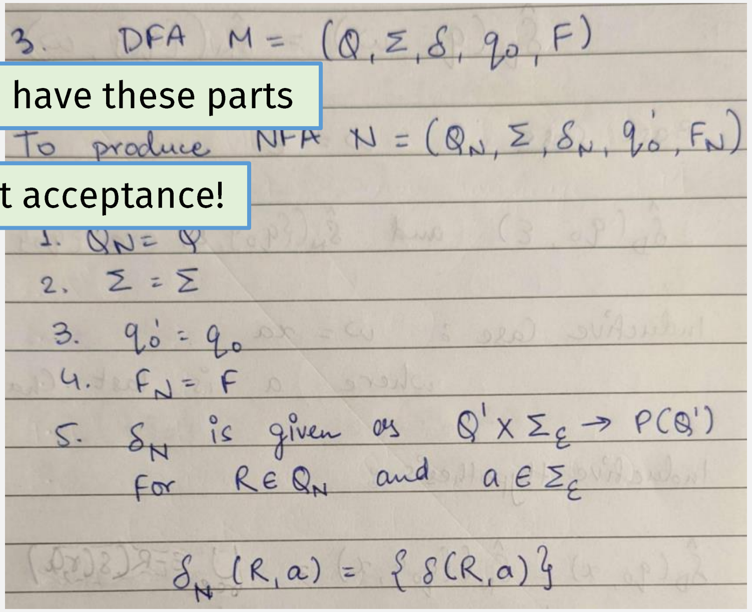
So correctness proof must also have these parts

<u>First assume</u>: $\hat{\delta}_N(q_0, w) = \{\hat{\delta}_M(q_0, w)\}$ ← This says nothing about acceptance!

- NOTE: This must match part 1's answer!
- Some invalid equalities:

$$\hat{\delta}_N(q_0, w) \neq \hat{\delta}_M(q_0, w)$$

$$\hat{\delta}_N(q_0, w) \neq \hat{\delta}_M(\{q_0\}, w)$$



HW2 Review

3 DFA->NFA

- Then prove that your procedure is correct, i.e., that M accepts some string w if and only if N accepts w. You'll probably want to use induction on the length of w.

Now prove: $\hat{\delta}_N(q_0, w) = \{\hat{\delta}_M(q_0, w)\}$

Proof: Using proof by induction on the length of string w

- **Base case:** We always start from the smallest string i.e., $w = \varepsilon$
  Applying this on the theorem, $\hat{\delta}_N(q0, \varepsilon)$ and $\{\hat{\delta}_M(q0, \varepsilon)\}$ we get $\{q0\}$ for both the cases. ← From definition of $\hat{\delta}$ (base case)

- **Inductive case:** For this we will take $w = xa$

  - **Inductive hypothesis:** $\hat{\delta}_N(q0, x) = \{\hat{\delta}_M(q0, x)\}$, call this set of states R
  - DFA last step from δ_M definition is given as $\{\delta_M(r, a)\}$ ← From definition of $\hat{\delta}$ (inductive case)
  - NFA last step from DFA → NFA definition is given as $\{\delta_M(r, a)\}$ ← From our NFA→DFA conversion

  Here, $r \in R$ and a is the last alphabet of the string w.

HW2 Review

5 A Closure Operation

Let EXPAND_c on a language L, where Σ is the alphabet of L and $c \in \Sigma$, be:

$$\text{EXPAND}_c(L) = \{wc \mid w \in L\}$$

Prove that, for any c, EXPAND_c is closed for regular languages.

Similar to closure proofs for union, concat, and star that we did in class

To prove that for any c, EXPAND_c is closed for regular languages, we need to create a DFA/NFA that recognizing it.

Extend L's NFA to recognize $\text{EXPAND}_c(L)$

L is regular so it must have an NFA recognizing it (thm from class)

Let $L = (Q_1, \Sigma_1, \delta_1, q_1, F_1)$, we construct $N = (Q, \Sigma, \delta, q_0, F)$ to recognize EXPAND_c

1. $Q = Q_1 \cup \{q_c\}$ where q_c is a new state appended to all the accept states of L with transition c.
2. The state q_0 is the same as the start state of L
3. The accept state F will be the new state $\{q_c\}$
4. Define δ so that for any $q \in Q$, and any $a \in \Sigma_\varepsilon$

$$\delta(q, a) = \begin{cases} \delta_1(q, a) & q \in Q_1 \text{ and } q \notin F \\ \delta_1(q, a) & q \in F_1 \text{ and } a \neq c \\ \{q_c\} & q \in F_1 \text{ and } a = c \end{cases}$$

$\text{EXPAND}_c(L)$ must be regular if it has an NFA recognizing it (thm from class)

Therefore EXPAND_c is closed for regular languages

Last Time:

Pumping lemma If A is a regular language, then there is a number p (the pumping length) where if s is any string in A of length at least p, then s may be divided into three pieces, $s = xyz$, satisfying the following conditions:

1. for each $i \geq 0$, $xy^iz \in A$,
2. $|y| > 0$, and
3. $|xy| \leq p$.

Let B be the language $\{0^n1^n | n \geq 0\}$. We use the pumping lemma to prove that B is not regular. The proof is by contradiction.

If this language is not regular, then what is it???

Maybe? … a **context-free language** (CFL)?

A Context-Free Grammar (CFG)

terminals

Top variable is
Start variable

$$A \to 0A1$$
$$A \to B$$
$$B \to \#$$

Variables
(also called a
nonterminals)

Substitution rules
(a.k.a., productions)

terminals (analogous to a DFA's alphabet)

CFGs: Formal Definition

A CFG Describes a Language!

grammar G_1

R is

Top variable is **Start variable**

terminals

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Variables (also called a nonterminal)

Substitution rules (a.k.a., productions)

terminals (analogous to a DFA's alphabet)

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the ***variables***,
2. Σ is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the start variable.

$V = \{A, B\}$,

$\Sigma = \{0, 1, \#\}$,

$S = A$,

Analogies

Regular Language	Context-Free Language (CFL)
Regular Expression	Context-Free Grammar (CFG)
A Reg expr describes a Regular lang	A CFG describes a CFL

Practical application:
Used to describe programming languages!

Java Language Described with CFGs

ORACLE

Java SE > Java SE Specifications > Java Language Specification

Chapter 2. Grammars

Prev

Chapter 2. Grammars

This chapter describes the context-free grammars used in this specification to define the lexical and syntactic structure of a program

2.1. Context-Free Grammars

A *context-free grammar* consists of a number of *productions*. Each production has an abstract symbol called a *nonterminal* as its *left-hand side*, and a sequence of one or more nonterminal and *terminal* symbols as its *right-hand side*. For each grammar, the terminal symbols are drawn from a specified *alphabet*.

Starting from a sentence consisting of a single distinguished nonterminal, called the *goal symbol*, a given context-free grammar specifies a language, namely, the set of possible sequences of terminal symbols that can result from repeatedly replacing any nonterminal in the sequence with a right-hand side of a production for which the nonterminal is the left-hand side.

2.2. The Lexical Grammar

A *lexical grammar* for the Java programming language is given in §3. This grammar has as its terminal symbols the characters of the Unicode character set. It defines a set of productions, starting from the goal symbol *Input* (§3.5), that describe how sequences of Unicode characters (§3.1) are translated into a sequence of input elements (§3.5)

https://docs.oracle.com/javase/specs/jls/se7/html/jls-2.html

(partially)

Python Language Described with a CFG

10. Full Grammar specification

This is the full Python grammar, as it is read by the parser generator and used to parse Python source files:

```
# Grammar for Python

# NOTE WELL: You should also follow all the steps listed at
# https://devguide.python.org/grammar/

# Start symbols for the grammar:
#       single_input is a single interactive statement;
#       file_input is a module or sequence of commands read from an input file;
#       eval_input is the input for the eval() functions.
#       func_type_input is a PEP 484 Python 2 function type comment
# NB: compound_stmt in single_input is followed by extra NEWLINE!
# NB: due to the way TYPE_COMMENT is tokenized it will always be followed by a NEWLINE
single_input: NEWLINE | simple_stmt | compound_stmt NEWLINE
file_input: (NEWLINE | stmt)* ENDMARKER
eval_input: testlist NEWLINE* ENDMARKER
```

(indentation checking probably not describable with a CFG)

https://docs.python.org/3/reference/grammar.html

~~Python~~ Many Language (partially) Described with a CFG

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```

https://docs.python.org/3/reference/grammar.html

Generating Strings with a CFG

A CFG Represents a Language!

$G_1 =$

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#$$

Strings in CFG's language
= all possible generated strings

$L(G_1)$ is $\{0^n\#1^n \mid n \geq 0\}$

Stop when string is all terminals

A CFG **generates** a string, by repeatedly applying substitution rules:

$$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$$

Start variable | After applying 1st rule | Use 1st rule | Use 1st rule | Use 2nd rule | Use last rule

Derivations: Formaly

A *context-free grammar* is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the ***variables***,
2. Σ is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the start variable.

Let $G = (V, \Sigma, R, S)$

Single-step

$$\alpha A \beta \underset{G}{\Rightarrow} \alpha \gamma \beta$$

Where:

$\alpha, \beta \in (V \cup \Sigma)^*$ ← Strings of terminals and variables

$A \in V$ ← Variable

$A \to \gamma \in R$ ← Rule

Extended Derivation

Base case: $\alpha \overset{*}{\underset{G}{\Rightarrow}} \alpha$

Recursive case:

- If $\alpha \overset{*}{\underset{G}{\Rightarrow}} \beta$ and $\beta \underset{G}{\Rightarrow} \gamma$
- Then: $\alpha \overset{*}{\underset{G}{\Rightarrow}} \gamma$

Formal Definition of a CFL

A ***context-free grammar*** is a 4-tuple (V, Σ, R, S), where

1. V is a finite set called the ***variables***,
2. Σ is a finite set, disjoint from V, called the ***terminals***,
3. R is a finite set of ***rules***, with each rule being a variable and a string of variables and terminals, and
4. $S \in V$ is the start variable.

$$G = (V, \Sigma, R, S)$$

$$L(G) = \left\{ w \in \Sigma^* \mid S \overset{*}{\underset{G}{\Rightarrow}} w \right\}$$

Any language that can be generated by some context-free grammar is called a ***context-free language***

Flashback: $\{0^n 1^n | \, n \geq 0\}$

- Pumping Lemma says it's not a regular language
- It's a context-free language!
  - Proof?
  - Come up with CFG describing it …
  - It's similar to:

$$A \rightarrow 0A1$$
$$A \rightarrow B$$
$$B \rightarrow \#\ \varepsilon$$

$L(G_1)$ is $\{0^n \# 1^n | \, n \geq 0\}$

$L = \{0^n1^n \mid n \geq 0\}$

rules of G:

$A \rightarrow 0A1 \mid \varepsilon$

Proof of Correctness

Correctness statement: $w \in L$ if and only if $A \overset{*}{\underset{G}{\Rightarrow}} w$

$\Rightarrow$ if $w \in L$ then $A \overset{*}{\underset{G}{\Rightarrow}} w$

Base case $w = \varepsilon$: if $\varepsilon \in L$ then $A \overset{*}{\underset{G}{\Rightarrow}} \varepsilon$ true, due to rule $A \rightarrow \varepsilon$

$\Leftarrow$ if $A \overset{*}{\underset{G}{\Rightarrow}} w$ then $w \in L$

Proof of Co[rrectness]

Note the parts of the proof:
- Clear and precise correctness statement
- All cases covered (⇒ and ⇐, base and inductive cases)
- Every step logically follows from previous
- Every step has a justification
- Uses the given facts (IH, etc)

$L = \{0^n 1^n \mid n \geq 0\}$

rules of G:

$A \rightarrow 0A1 \mid \varepsilon$

Correctness statement: $w \in L$ if and only if $A \overset{*}{\underset{G}{\Rightarrow}} w$

⇒ if $w \in L$ then $A \overset{*}{\underset{G}{\Rightarrow}} w$

Base case $w = \varepsilon$: if $\varepsilon \in L$ then $A \overset{*}{\underset{G}{\Rightarrow}} \varepsilon$ true, due to rule $A \rightarrow \varepsilon$

Inductive case $w = 0x1$ (odd length strings not in L!)

IH: if $x \in L$ then $A \overset{*}{\underset{G}{\Rightarrow}} x$

Need to prove: if $0x1 \in L$ then $A \overset{*}{\underset{G}{\Rightarrow}} 0x1$

if $0x1 \in L$ then $x \in L$ (def of L) and $A \overset{*}{\underset{G}{\Rightarrow}} x$ (by IH)

if $A \overset{*}{\underset{G}{\Rightarrow}} x$ then $A \overset{*}{\underset{G}{\Rightarrow}} 0x1$, by def of $\overset{*}{\underset{G}{\Rightarrow}}$ and rule $A \rightarrow 0A1$

Therefore: if $0x1 \in L$ then $A \overset{*}{\underset{G}{\Rightarrow}} 0x1$

⇐ if $A \overset{*}{\underset{G}{\Rightarrow}} w$ then $w \in L$

HW4

A String Can Have Multiple Derivations

$$\langle \text{EXPR} \rangle \rightarrow \langle \text{EXPR} \rangle + \langle \text{TERM} \rangle \mid \langle \text{TERM} \rangle$$
$$\langle \text{TERM} \rangle \rightarrow \langle \text{TERM} \rangle \times \langle \text{FACTOR} \rangle \mid \langle \text{FACTOR} \rangle$$
$$\langle \text{FACTOR} \rangle \rightarrow (\langle \text{EXPR} \rangle) \mid \texttt{a}$$

String to generate: **a + a × a**

- EXPR ⇒
- EXPR **+** TERM ⇒
- EXPR **+** TERM × FACTOR ⇒
- EXPR **+** TERM × **a** ⇒

  …

RIGHTMOST DERIVATION

- EXPR ⇒
- EXPR **+** TERM ⇒
- TERM **+** TERM ⇒
- FACTOR **+** TERM ⇒
- **a** **+** TERM

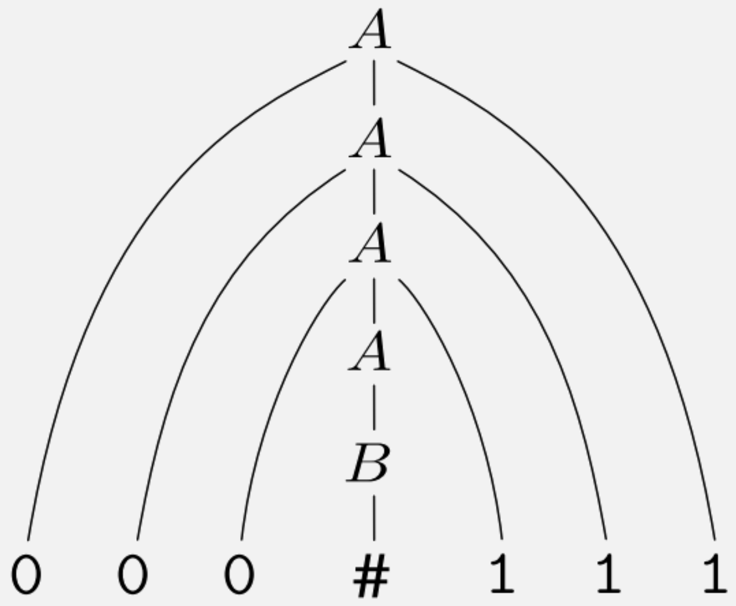
  …

LEFTMOST DERIVATION

Derivations and Parse Trees

$A \Rightarrow 0A1 \Rightarrow 00A11 \Rightarrow 000A111 \Rightarrow 000B111 \Rightarrow 000\#111$

A derivation may also be represented as a **parse tree**



Multiple Derivations, Single Parse Tree

Leftmost deriviation

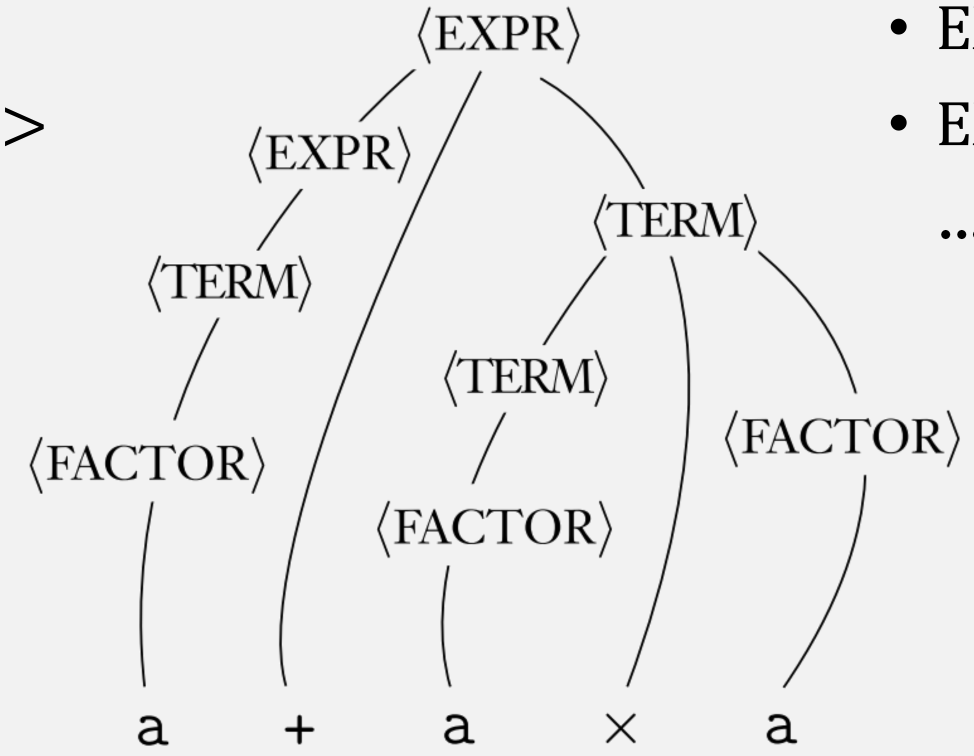
- EXPR =>
- EXPR + TERM =>
- TERM + TERM =>
- FACTOR + TERM =>
- a + TERM

…

Rightmost deriviation

- EXPR =>
- EXPR + TERM =>
- EXPR + TERM x FACTOR =>
- EXPR + TERM x a=>

…



A Parse Tree gives "meaning" to a string

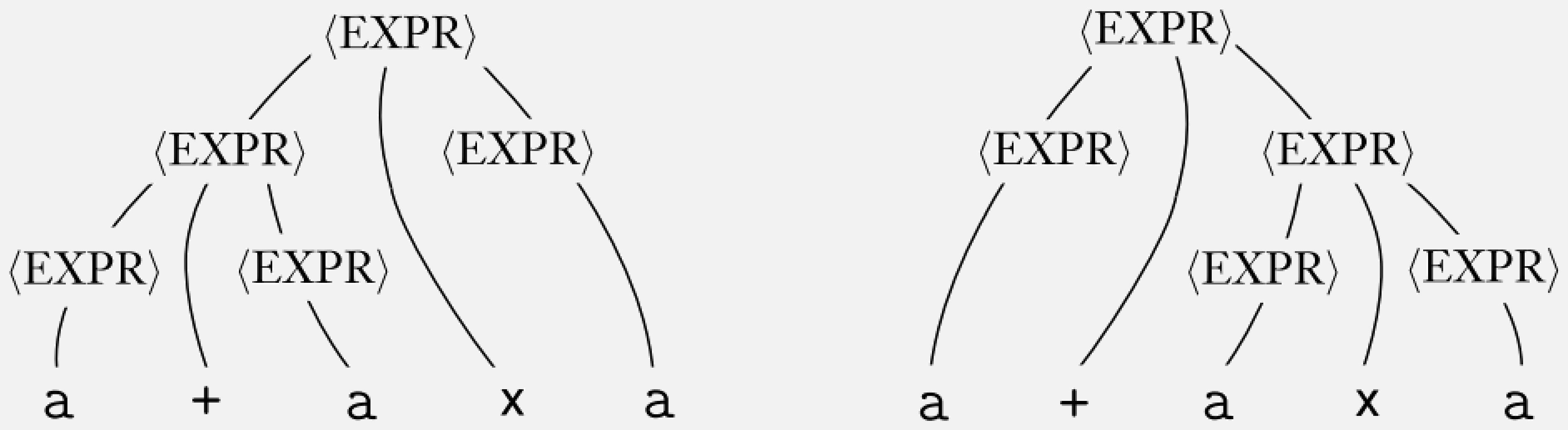
Since the "meaning" (i.e., parse tree) is same, by convention we just use **leftmost** derivation

Ambiguity

grammar G_5:

$$\langle \text{EXPR} \rangle \rightarrow \langle \text{EXPR} \rangle + \langle \text{EXPR} \rangle \mid \langle \text{EXPR} \rangle \times \langle \text{EXPR} \rangle \mid (\langle \text{EXPR} \rangle) \mid a$$

Same string,
Different derivation,
<u>and</u> different parse tree!



Ambiguity

A string w is derived ***ambiguously*** in context-free grammar G if it has two or more different leftmost derivations. Grammar G is ***ambiguous*** if it generates some string ambiguously.

An ambiguous grammar can give a string multiple meanings! (why is this **bad**?)

Real-life Ambiguity ("Dangling" else)

- What is the result of this C program?
  - `if (1) if (0) printf("a"); else printf("2");`

```
if (1)
  if (0)
    printf("a");
  else
    printf("2");
```

VS

```
if (1)
  if (0)
    printf("a");
else
  printf("2");
```

Ambiguous grammars are confusing. In a language, a string (program) should have only **one meaning**.

Problem is, there's no guaranteed way to create an unambiguous grammar (so language designers must be careful)

Designing Grammars : Basics

- Think about what you want to “link” together
- E.g., XML
  - ELEMENT → <TAG>CONTENT</TAG>
  - Start and end tags are “linked”

- Start with small grammars and then combine (just like FSMs)

Designing Grammars: Building Up

- Start with small grammars and then combine (just like FSMs)
  - To create a grammar for the language $\{0^n 1^n | n \geq 0\} \cup \{1^n 0^n | n \geq 0\}$
  - First create grammar for lang $\{0^n 1^n | \, n \geq 0\}$:

$$S_1 \rightarrow 0S_1 1 \mid \varepsilon$$

  - Then create grammar for lang $\{1^n 0^n | \, n \geq 0\}$:

$$S_2 \rightarrow 1S_2 0 \mid \varepsilon$$

  - Then combine: $S \rightarrow S_1 \mid S_2$

$$S_1 \rightarrow 0S_1 1 \mid \varepsilon$$
$$S_2 \rightarrow 1S_2 0 \mid \varepsilon$$

"|" = "or" = union
(combines 2 rules with same left side)

Closed Operations on CFLs

- Start with small grammars and then combine (just like FSMs)
- "Or": $S \to S_1 \mid S_2$
- "Concatenate": $S \longrightarrow S_1 S_2$
- "Repetition": $S' \longrightarrow S' S_1 \mid \varepsilon$

In-class exercise: Designing grammars

alphabet Σ is $\{0,1\}$

$\{w \mid w \text{ starts and ends with the same symbol}\}$

- S -> $0C'0$ | $1C'1$ | ε — “string starts/ends with same symbol, middle can be anything”
- C' -> $C'C$ | ε — “all possible terminals, repeated (ie, all possible strings)”
- C -> 0 | 1 — “all possible terminals”

Check-in Quiz 9/29

On gradescope