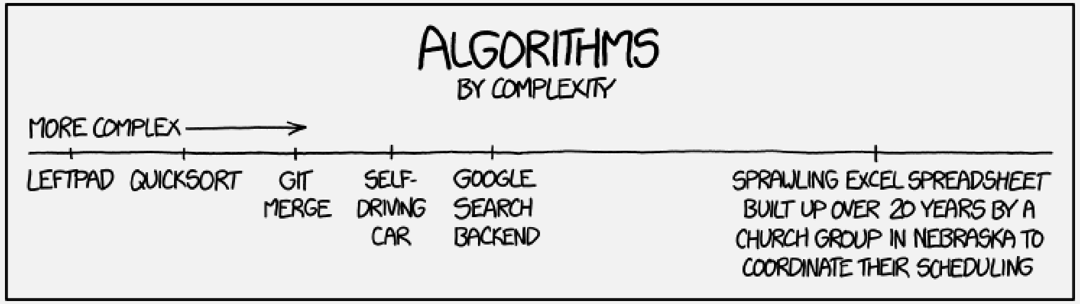
UMB CS622

Time Complexity

Monday, November 8, 2021

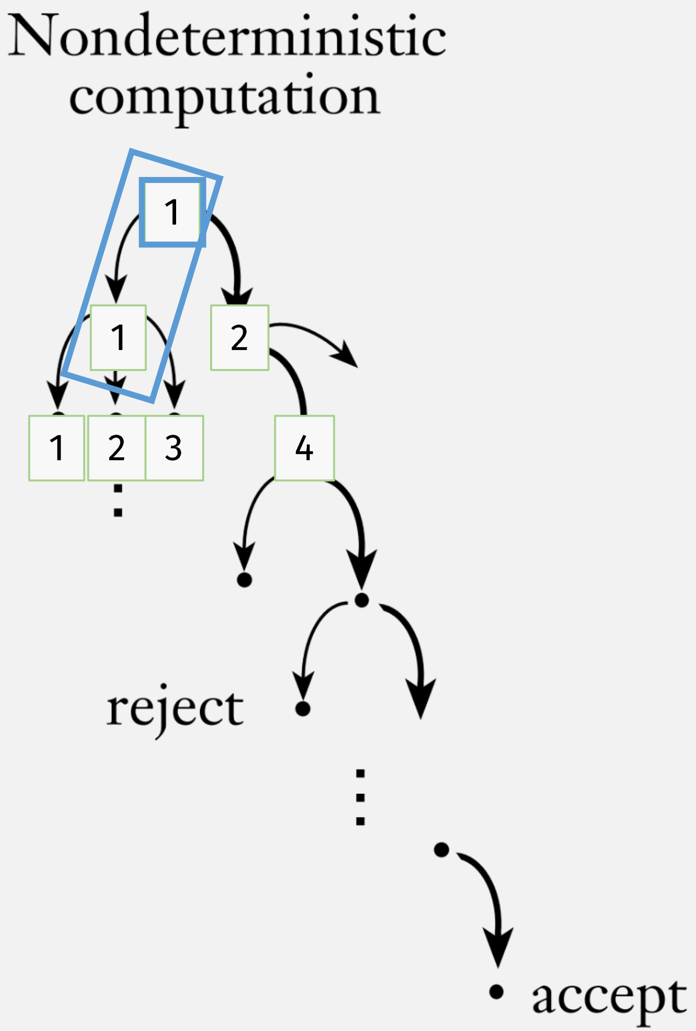


Announcements

- HW7 due Wed 11:59pm EST
- Submit “HW Solution Plans” to Piazza
  - Not at the last minute please
- HW5 grades returned

Flashback: Nondet. TM → Deterministic TM

- To simulate NTM with Det. TM:
  - Number the nodes at each step
  - Deterministically check every tree path, in breadth-first order
    - Root node: 1
    - **1-1**



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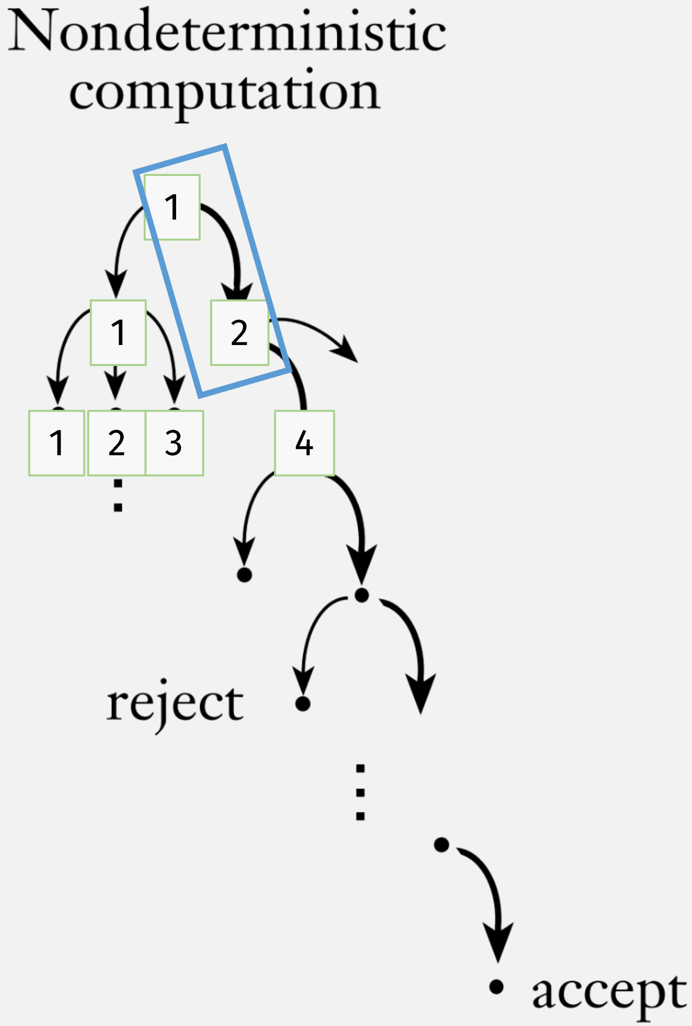

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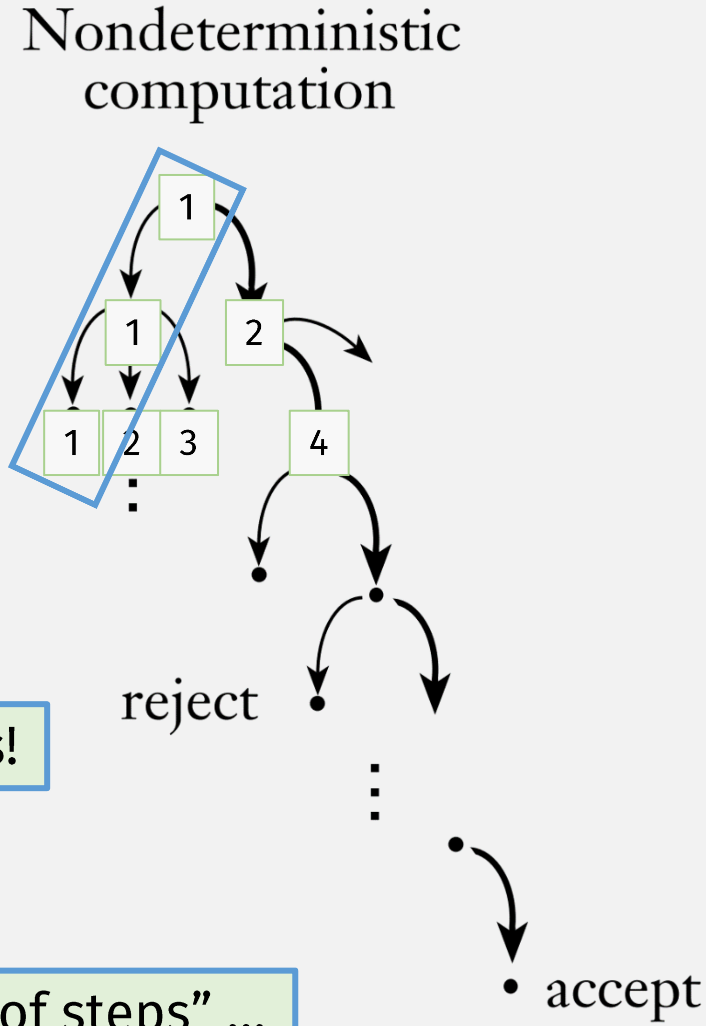

A TM and a NTM are "equivalent" …

.. but not if we care about the # of steps!

So how inefficient is it?

First, we need a formal way to count "# of steps" …

A Simpler Example: $A = \{0^k 1^k | k \geq 0\}$

M_1 = "On input string w:

1. Scan across the tape and *reject* if a 0 is found to the right of a 1.
2. Repeat if both 0s and 1s remain on the tape:
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4. If 0s still remain after all the 1s have been crossed off, or if 1s still remain after all the 0s have been crossed off, *reject*. Otherwise, if neither 0s nor 1s remain on the tape, *accept*."

# of steps (worst case), ***n*** **= length of input:**

➢ TM Line 1:

- ***n*** steps to scan + ***n*** steps to return to beginning = ***2n* steps**

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# of steps (worst case), n = length of input:

- TM Line 1:
  - n steps to scan + n steps to return to beginning = **$2n$ steps**
- Lines 2-3 (loop):
  - steps/iteration (line 3): $n/2$ steps to find "1" + $n/2$ steps to return = n steps
  - # iterations (line 2): Each scan crosses off 2 chars, so at most $n/2$ scans
  - Total = steps/iteration * # iterations = $n(n/2)$ = **$n^2/2$ steps**

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$\boxed{n^2/2 + 3n}$

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- Lines 2-3 (loop):
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  - # iterations (line 2): Each scan crosses off 2 chars, so at most $n/2$ scans
  - Total = steps/iteration * # iterations = $n\,(n/2)$ = $\boxed{n^2/2 \text{ steps}}$
- Line 4:
  - $\boxed{n \text{ steps}}$ to scan input one more time
- Total: $\boxed{2n + n^2/2 + n} = \boxed{n^2/2 + 3n \text{ steps}}$

Interlude: Polynomials

order/degree

coefficients

Highest order term

$$\underline{6n^3} + \underline{2n^2} + \underline{20n} + \underline{45}$$

terms

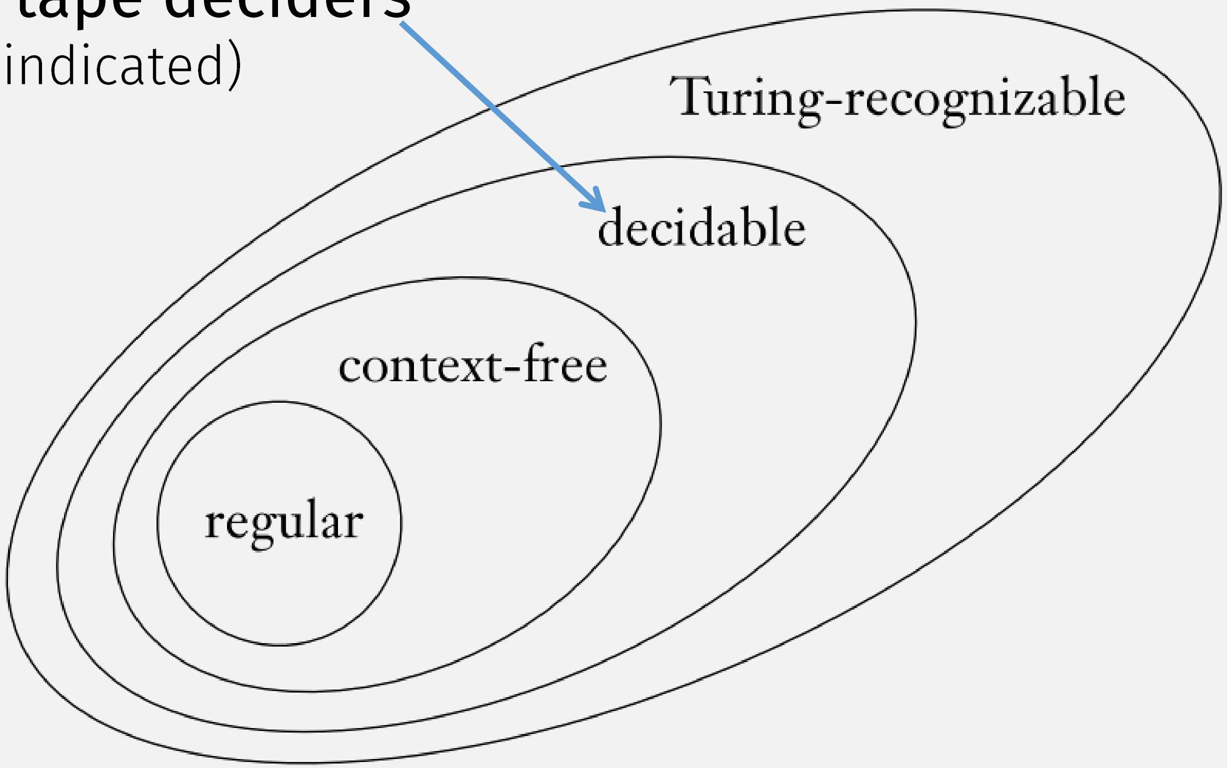
Definition: Time Complexity

i.e., a decider (algorithm)

Let M be a deterministic Turing machine that halts on all inputs. The ***running time*** or ***time complexity*** of M is the function $f\colon \mathcal{N} \longrightarrow \mathcal{N}$, where $f(n)$ is the maximum number of steps that M uses on any input of length n. If $f(n)$ is the running time of M, we say that M runs in time $f(n)$ and that M is an $f(n)$ time Turing machine. Customarily we use n to represent the length of the input.

Where Are We Now?

We are back in here now:
deterministic, single-tape deciders
(unless otherwise indicated)



Definition: Time Complexity

NOTE: n has no units, it's only *roughly* "length" of the input

n can be:
characters,
states,
nodes, ...

We can use any n that is correlated with the input length

Let M be a deterministic Turing machine that halts on all inputs. The ***running time*** or ***time complexity*** of M is the function $f: \mathcal{N} \longrightarrow \mathcal{N}$, where $f(n)$ is the maximum number of steps that M uses on any input of length n. If $f(n)$ is the running time of M, say that M runs in time $f(n)$ and that M is an $f(n)$ time Tur- machine. Customarily we use n to represent the length of the ut.

- Machine M_1 that decides $A = \{0^k1^k | k \geq 0\}$
  - Running time / Time Complexity: $\boldsymbol{n^2/2+3n}$

M_1 = "On input string w:
1. Scan across the tape and *reject* if a 0 is found to the right of a 1.
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Interlude: Asymptotic Analysis

Total: $\boldsymbol{n^2 + 3n}$

- If $\boldsymbol{n} = 1$
  - $\boldsymbol{n^2} = 1$
  - $\boldsymbol{3n} = 3$
  - Total = 4
- If $\boldsymbol{n} = 10$
  - $\boldsymbol{n^2} = 100$
  - $\boldsymbol{3n} = 30$
  - Total = 130
- If $\boldsymbol{n} = 100$
  - $\boldsymbol{n^2} = 10{,}000$
  - $\boldsymbol{3n} = 300$
  - Total = 10,300
- If $\boldsymbol{n} = 1{,}000$
  - $\boldsymbol{n^2} = 1{,}000{,}000$
  - $\boldsymbol{3n} = 3{,}000$
  - Total = 1,003,000

asymptotic analysis only cares about **large** n

$\boldsymbol{n^2 + 3n} \approx \boldsymbol{n^2}$ as n gets large

Definition: Big-O Notation

Let f and g be functions $f, g: \mathcal{N} \longrightarrow \mathcal{R}^+$. Say that $\boxed{\boldsymbol{f(n) = O(g(n))}}$ if positive integers c and n_0 exist such that for $\boxed{\text{every integer } n \geq n_0}$,

$$\boxed{f(n) \leq c\,g(n).}$$

“only care about **large** n“

When $f(n) = O(g(n))$, we say that $g(n)$ is an ***upper bound*** for $f(n)$, or more precisely, that $\boxed{g(n) \text{ is an } \textbf{\textit{asymptotic upper bound}}}$ for $f(n)$, to emphasize that we are suppressing constant factors.

In other words: Keep only highest order term, drop all coefficients

- Machine M_1 that decides $A = \{0^k1^k | k \geq 0\}$
  - Is an $n^2 + 3n$ time Turing machine
  - Is an $O(n^2)$ time Turing machine
  - Has asymptotic upper bound $O(n^2)$

Definition: Small-o Notation (less used)

Let f and g be functions $f, g \colon \mathcal{N} \longrightarrow \mathcal{R}^+$. Say that $\boldsymbol{f(n) = o(g(n))}$ if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$$

In other words, $f(n) = o(g(n))$ means that for any real number $c > 0$, a number n_0 exists, where $\boxed{f(n) < c\,g(n)}$ for all $n \geq n_0$.

Analogy: Big-O : ≤ :: small-o : <

Let f and g be functions $f, g \colon \mathcal{N} \longrightarrow \mathcal{R}^+$. Say that $\boldsymbol{f(n) = O(g(n))}$ if positive integers c and n_0 exist such that for every integer $n \geq n_0$,

$$\boxed{f(n) \leq c\,g(n).}$$

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Big-O arithmetic

- $O(\boldsymbol{n^2}) + O(\boldsymbol{n^2})$
  $= O(\boldsymbol{n^2})$

- $O(\boldsymbol{n^2}) + O(\boldsymbol{n})$
  $= O(\boldsymbol{n^2})$

- $\boldsymbol{2n} = O(\boldsymbol{n})$?
  - TRUE

- $\boldsymbol{2n} = O(\boldsymbol{n^2})$?
  - TRUE

- $\boldsymbol{1} = O(\boldsymbol{n^2})$?
  - TRUE

- $\boldsymbol{2^n} = O(\boldsymbol{n^2})$?
  - FALSE

Definition: Time Complexity Classes

Let $t\colon \mathcal{N} \longrightarrow \mathcal{R}^+$ be a function. Define the ***time complexity class***, $\mathbf{TIME}(\boldsymbol{t(n)})$, to be the collection of all languages that are decidable by an $O(t(n))$ time Turing machine.

Remember: **TMs** have a time complexity (i.e., a running time), **languages** are in a time complexity class

The complexity class of a **language** is determined by the time complexity (running time) of its deciding **TM**

- Machine M_1 decides language $A = \{0^k 1^k \mid k \geq 0\}$
  - M_1 has time complexity (running time) of $O(\boldsymbol{n^2})$
  - A is in time complexity class $\mathbf{TIME}(\boldsymbol{n^2})$

A Faster Machine? $A = \{0^k 1^k \mid k \geq 0\}$

$M_2 =$ "On input string w:

1. Scan across the tape and *reject* if a 0 is found to the right of a 1.
2. Repeat as long as some 0s and some 1s remain on the tape:
3. Scan across the tape, checking whether the total number of 0s and 1s remaining is even or odd. If it is odd, *reject*.
4. Scan again across the tape, crossing off every other 0 starting with the first 0, and then crossing off every other 1 starting with the first 1.
5. If no 0s and no 1s remain on the tape, *accept*. Otherwise, *reject*."

Previously:

$M_1 =$ "On input string w:

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Number of steps (worst case), n = **length of input:**

- Line 1:
  - $\boldsymbol{n}$ steps to scan + $\boldsymbol{n}$ steps to return to beginning = $O(\boldsymbol{n})$ steps

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Number of steps (worst case), n = **length of input:**

- Line 1:
  - n steps to scan + n steps to return to beginning = $O(n)$ steps

➢ Lines 2-4 (loop):
  - **steps/iteration** (lines 3-4): a scan takes $O(n)$ steps
  - **# iters** (line 2): Each iter crosses off *half* the chars, so at most $O(\log n)$ scans
  - Total: $O(n) * O(\log n)$ = $O(n \log n)$ steps

Interlude: Logarithms

- If $2^x = y$ …
- … then $\log_2 y = x$
- $\log_2 n = O(\mathbf{log}\ \boldsymbol{n})$
  - "divide and conquer" algorithms = $O(\mathbf{log}\ \boldsymbol{n})$
  - E.g., binary search
- (In computer science, **base-2 is the only base!**)

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Number of steps (worst case), n = **length of input:**

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- Lines 2-4 (loop):
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  - # iters (line 2): Each iter crosses off *half* the chars, so at most $O(\log \boldsymbol{n})$ scans
  - Total: $O(\boldsymbol{n})$ * $O(\log \boldsymbol{n})$ = $O(\boldsymbol{n} \log \boldsymbol{n})$ steps
- ➢ Line 5:
  - $O(\boldsymbol{n})$ steps to scan input one more time

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$\boxed{O(\boldsymbol{n} \log \boldsymbol{n})}$

Prev: $\boldsymbol{n^2/2 + 3n = O(n^2)}$

Number of steps (worst case), n = **length of input:**

- Line 1:
  - $\boldsymbol{n}$ steps to scan + $\boldsymbol{n}$ steps to return to beginning = $\boxed{O(\boldsymbol{n}) \text{ steps}}$
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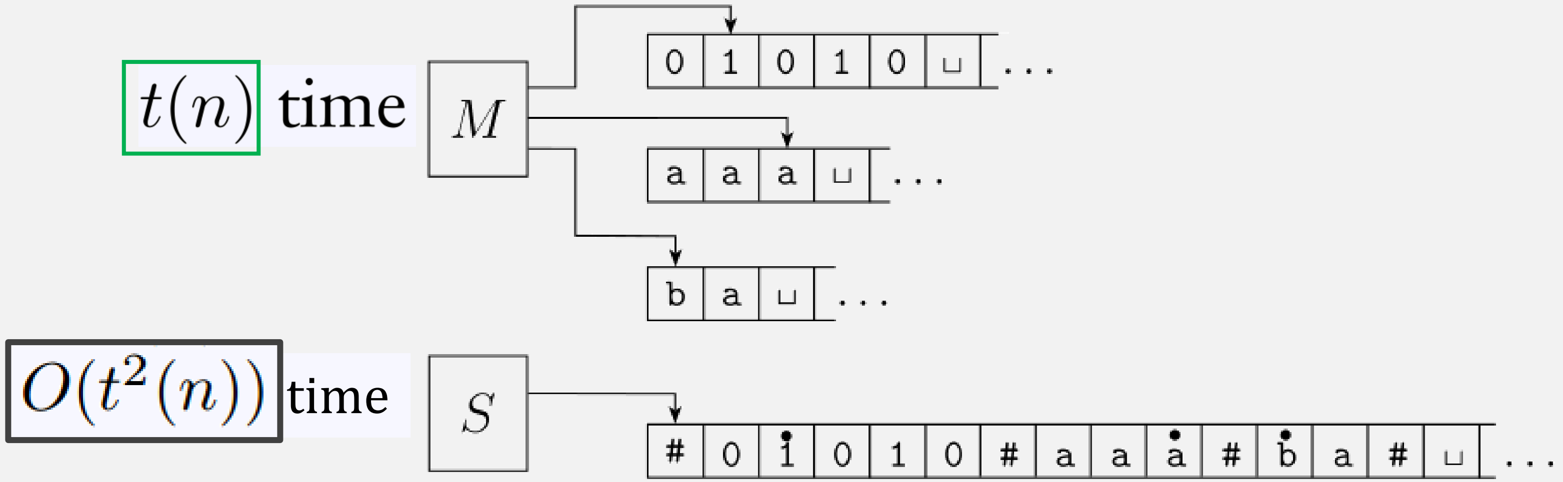
  - # iters (line 2): Each iter crosses off *half* the chars, so at most $O(\log \boldsymbol{n})$ scans
  - Total: $O(\boldsymbol{n}) * O(\log \boldsymbol{n}) = \boxed{O(\boldsymbol{n} \log \boldsymbol{n})}$ steps
- Line 5:
  - $\boxed{O(\boldsymbol{n}) \text{ steps}}$ to scan input one more time
- Total: $\boxed{O(\boldsymbol{n}) + O(\boldsymbol{n} \log \boldsymbol{n}) + O(\boldsymbol{n})}$ =

Terminology: Categories of Bounds

- Exponential time
  - $O(\mathbf{2}^{\boldsymbol{n}\text{^}\boldsymbol{c}})$, for $c > 0$, or $\mathbf{2}^{O(\boldsymbol{n})}$ (always base 2)
- Polynomial time
  - $O(\boldsymbol{n}^{\boldsymbol{c}})$, for $c > 0$
- Quadratic time (special case of polynomial time)
  - $O(\boldsymbol{n}^{\mathbf{2}})$
- Linear time (special case of polynomial time)
  - $O(\boldsymbol{n})$
- Log time
  - $O(\mathbf{log}\,\boldsymbol{n})$

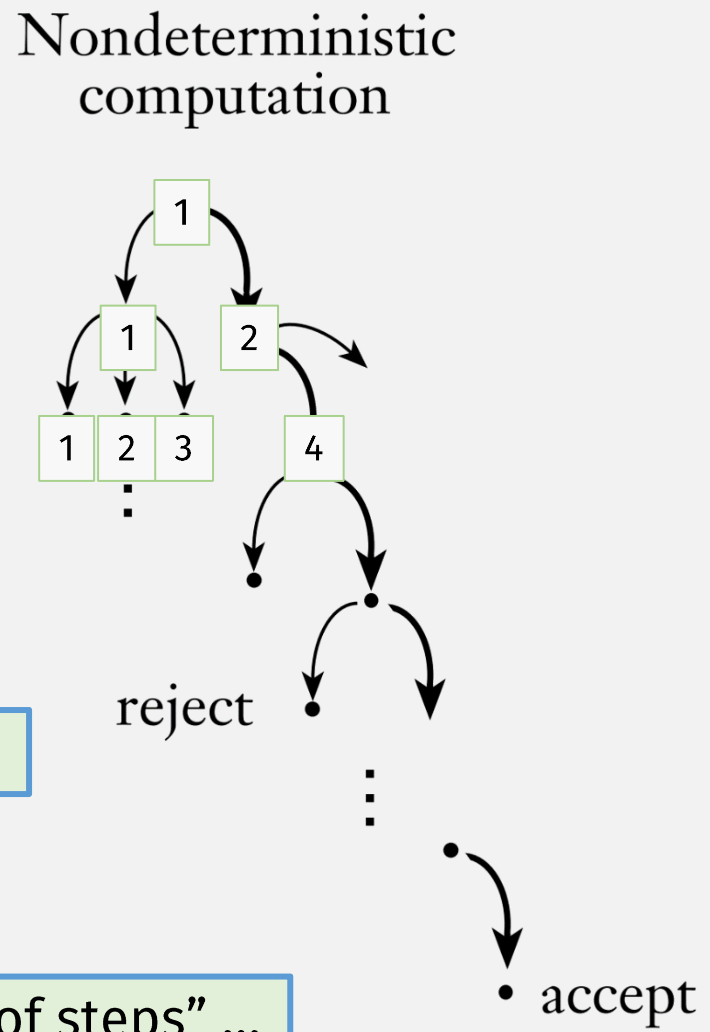
Multi-tape vs Single-tape TMs: # of Steps



- For single-tape TM to simulate 1 step of multi-tape:
  - Scan to find all "heads" = O(length of all M's tapes)
  - "Execute" transition at all the heads = O(length of all M's tapes)
- # single-tape steps to simulate 1 multitape step (worst case)
  - = O(length of all M's tapes)
  - = $O(t(n))$, If M spends all its steps expanding its tapes
- <u>Total steps (single tape)</u>: $O(t(n))$ per step × $t(n)$ steps =

Flashback: Nondet. TM → Deterministic TM

- Simulate NTM with Det. TM:
  - Number the nodes at each step
  - Deterministically check every tree path, in breadth-first order
    - 1
    - 1-1
    - 1-2
    - **1-1-1**



A TM and a NTM are "equivalent" ...

.. but not if we care about the # of steps

How inefficient is it?

First, we need a formal way to count "# of steps" ...

Flashback: Nondet. TM → Deterministic TM

$t(n)$ time $\boxed{2^{O(t(n))}}$ time

- Simulate NTM with Det. TM:
  - Number the nodes at each step
  - Deterministically check every tree path, in breadth-first order
    - 1
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    - 1-2
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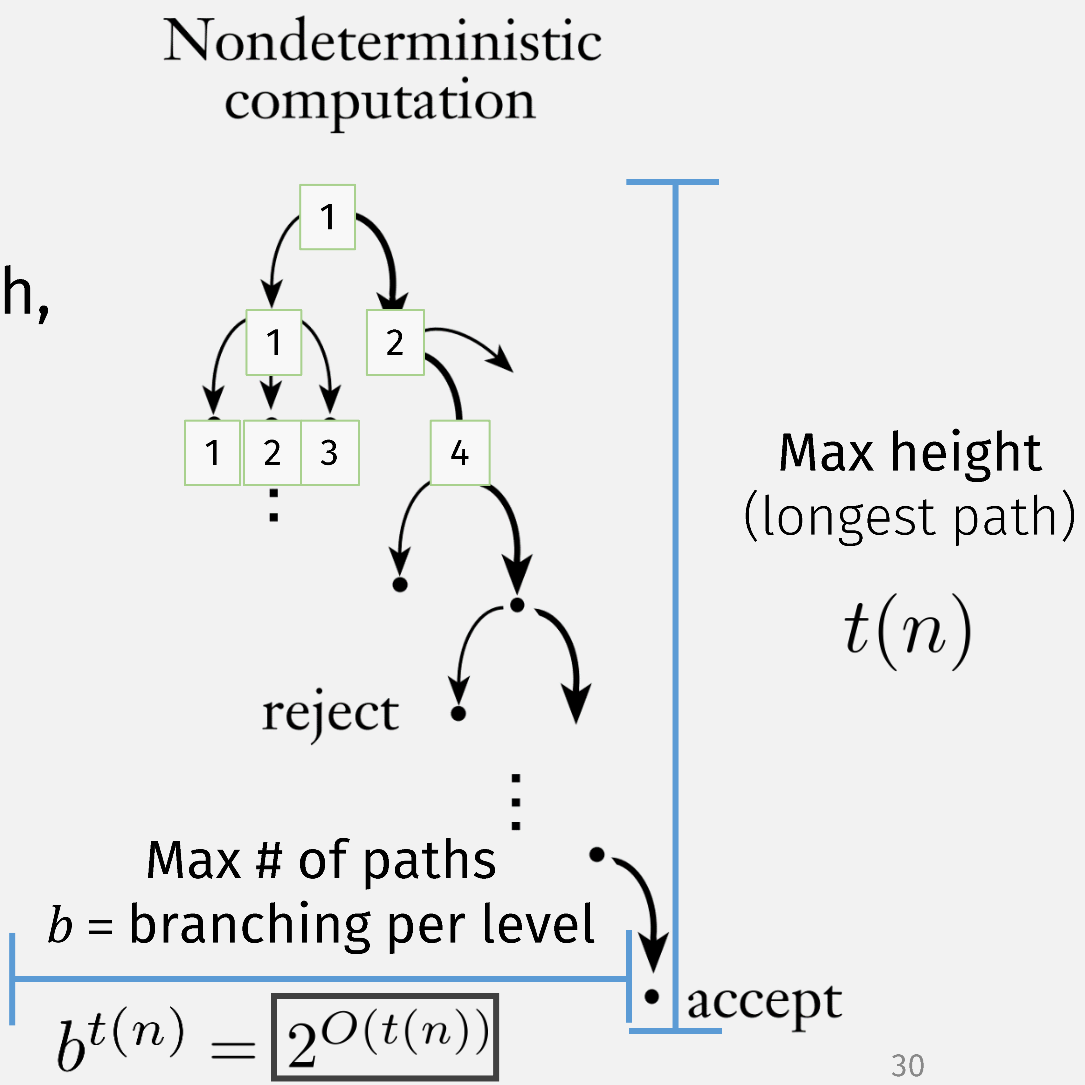
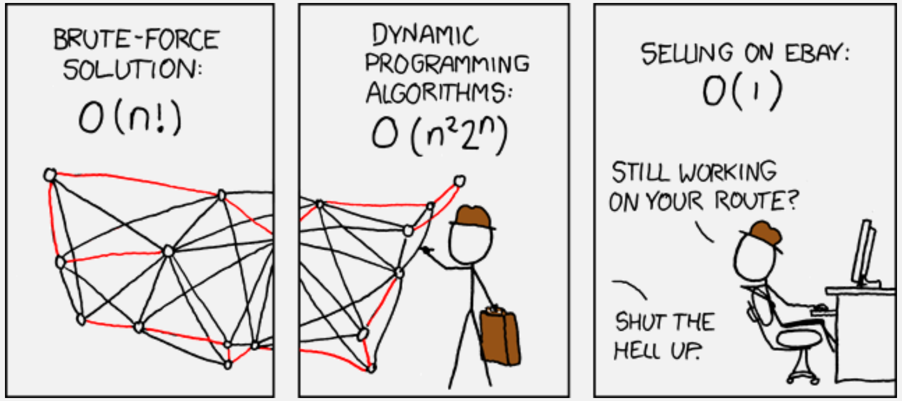


Summary: TM Variations

- If multi-tape TM: $t(n)$ time
- Then equivalent single-tape TM: $O(t^2(n))$
  - **Quadratically** slower

- If non-deterministic TM: $t(n)$ time
- Then equivalent single-tape TM: $2^{O(t(n))}$
  - **Exponentially** slower

Polynomial Time (P)



The Polynomial Time Complexity Class (**P**)

P is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$P = \bigcup_k \mathrm{TIME}(n^k).$$

- Corresponds to "realistically" solvable problems:
  - Problems in **P** = "solvable" or "tractable"
  - Problems outside **P** = "unsolvable" or "intractable"

“Unsolvable” Problems

Mathematicians are weird.

- Unsolvable problems (those outside **P**):
  - usually only have “brute force” solutions
  - i.e., “try all possible inputs”
  - “unsolvable” applies only to large n

Brute-force attack

From Wikipedia, the free encyclopedia

In cryptography, a **brute-force attack** consists of an attacker submitting many passwords or passphrases with the hope of eventually guessing a combination correctly. The attacker systematically checks all possible passwords and passphrases until the correct one is found. Alternatively, the attacker can attempt to guess the key which is typically created from the password using a key derivation function. This is known as an **exhaustive key search**.

Amount of Time to Crack Passwords

“abcdefg” 7 characters	.29 milliseconds
“abcdefgh” 8 characters	5 hours
“abcdefghi” 9 characters	5 days
“abcdefghij” 10 characters	4 months
“abcdefghijk” 11 characters	1 decade
“abcdefghijkl” 12 characters	2 centuries

Do these problems exist???

3 Problems in $\mathbf{P}$

- A Graph Problem:

  $PATH = \{\langle G, s, t\rangle \mid G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$

- A Number Problem:

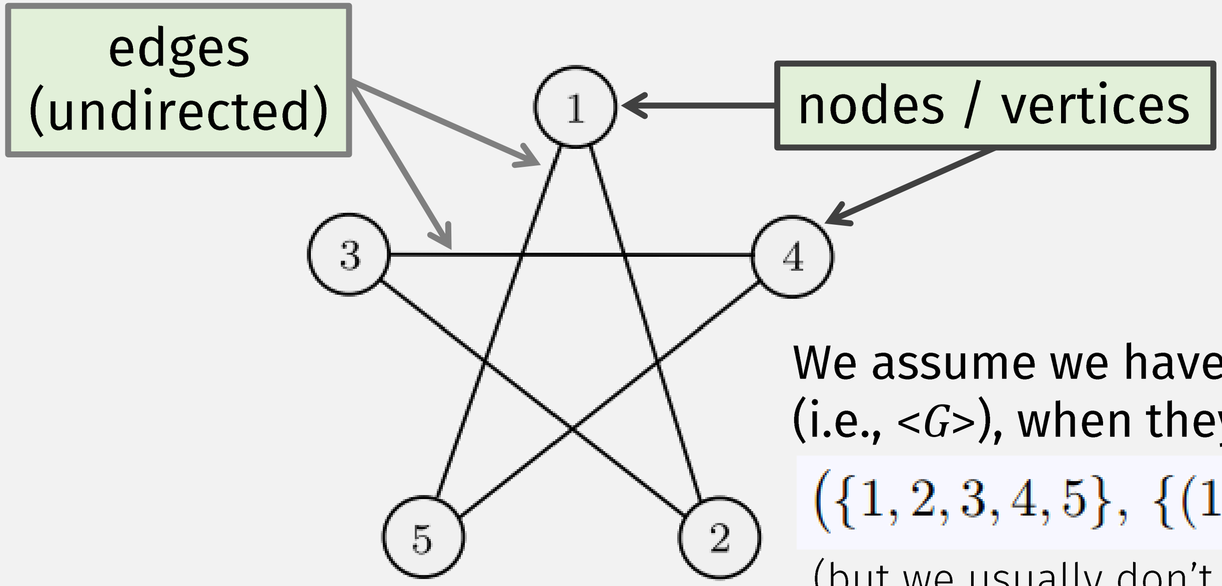
  $RELPRIME = \{\langle x, y\rangle \mid x \text{ and } y \text{ are relatively prime}\}$

- A CFL Problem:

  Every context-free language is a member of P

- To prove that a language is in $\mathbf{P}$ ...
  - ... we construct a polynomial time algorithm deciding the language
- (These also have nonpolynomial, i.e., brute force, algorithms)
  - Check all possible ... paths/numbers/strings ...

Interlude: Graphs (see Sipser Chapter 0)



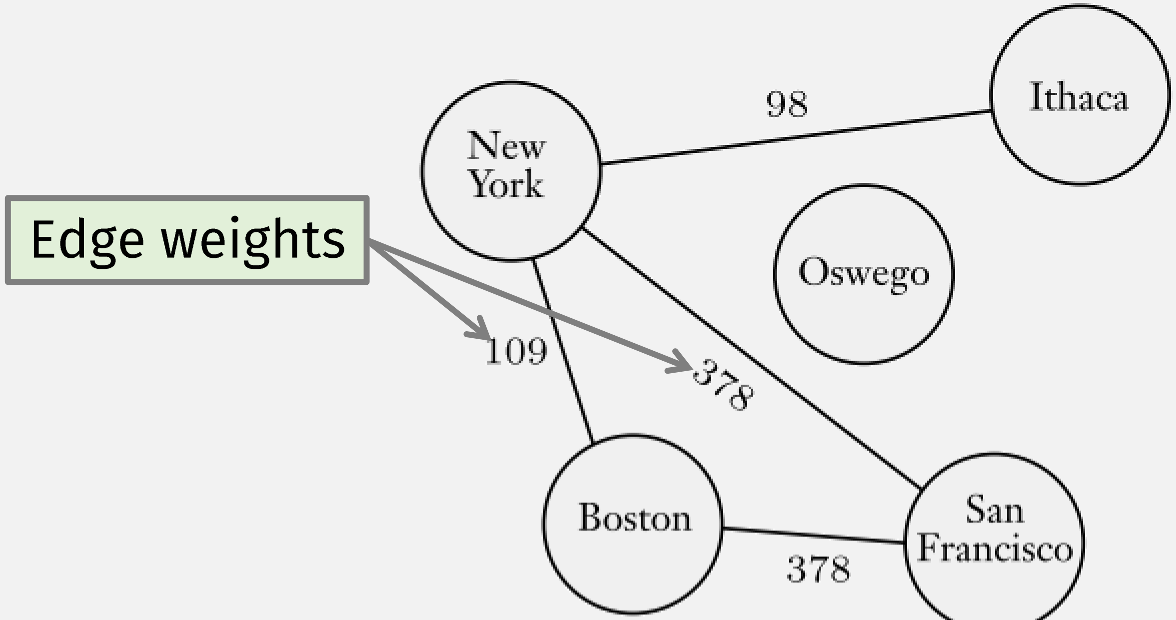
We assume we have *some* **string encoding of a graph** (i.e., <G>), when they are args to TMs, e.g.:

$$(\{1,2,3,4,5\},\ \{(1,2),\ (2,3),\ (3,4),\ (4,5),\ (5,1)\})$$

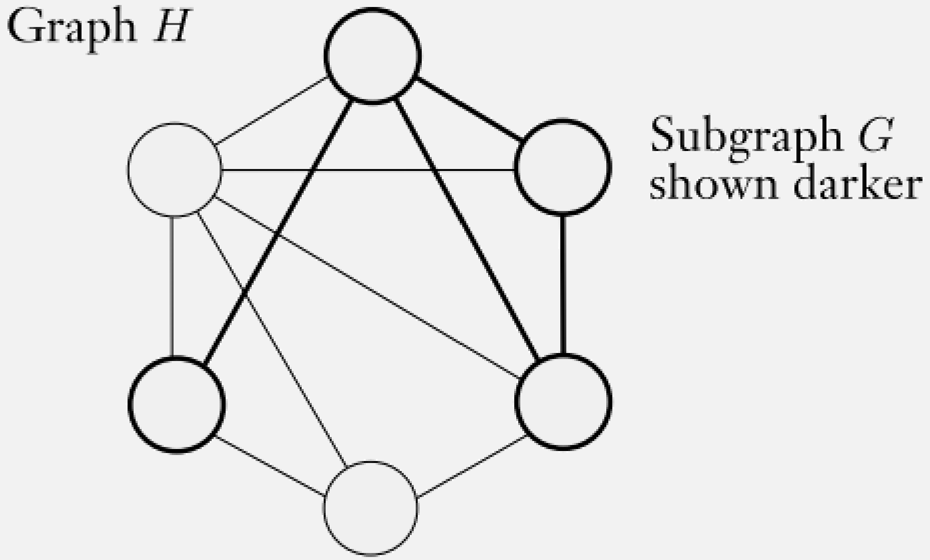
(but we usually don't care about the actual details)

- Edge defined by two nodes (order doesn't matter)
- Formally, a graph = a pair (V, E)
  - Where V = a set of nodes, E = a set of edges

Interlude: Weighted Graphs

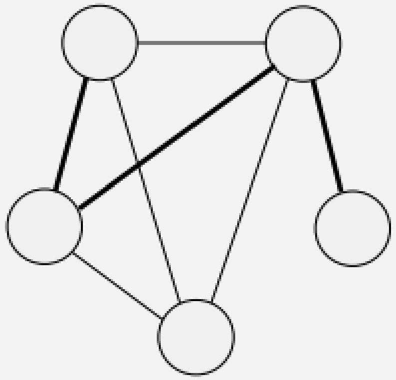


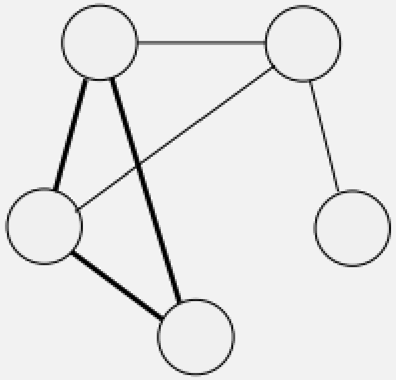
Interlude: Subgraphs

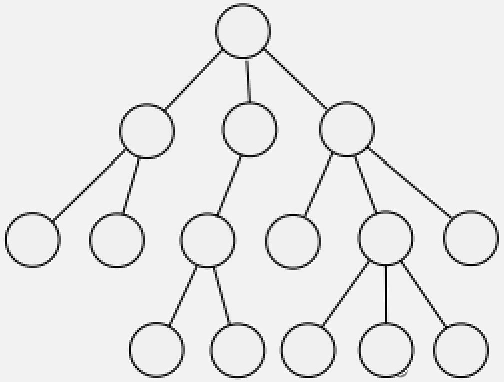


Interlude: Paths and other Graph Things

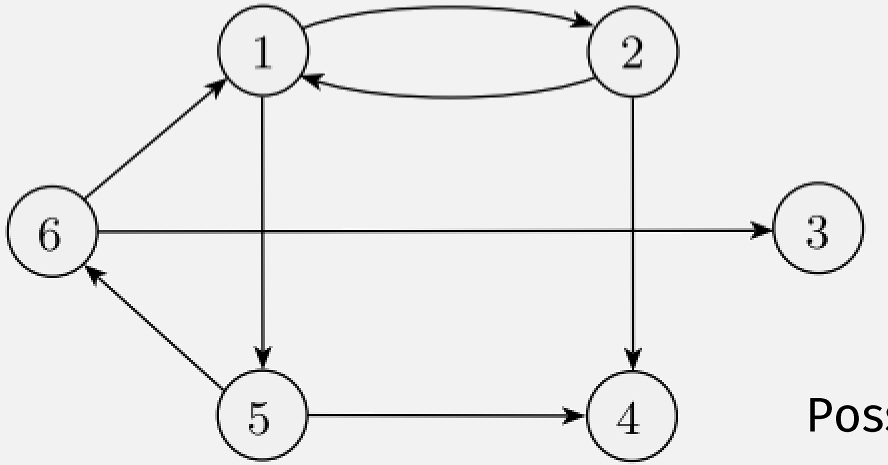
- Path
  - A sequence of nodes connected by edges
- Cycle
  - A path that starts/ends at the same node
- Connected graph
  - Every two nodes has a path
- Tree
  - A connected graph with no cycles







Interlude: Directed Graphs



Possible **string encoding** given to TMs:

$(\{1,2,3,4,5,6\},\ \{(1,2), (1,5), (2,1), (2,4), (5,4), (5,6), (6,1), (6,3)\})$

Each pair of nodes included twice

- Directed graph = (V, E)
  - V = set of nodes, E = set of edges
- An edge is a pair of nodes (u,v), **order now matters**
  - u = "from" node, v = "to" node
- "degree" of a node: number of edges connected to the node
  - Nodes in a directed graph have both indegree and outdegree

Interlude: Graph Encodings

$(\{1,2,3,4,5\},\ \{(1,2),\ (2,3),\ (3,4),\ (4,5),\ (5,1)\})$

- For graph algorithms, "length of input" n is usually # of vertices
  - (Not number of chars in the encoding)
- So given graph $G = (V, E)$, $n = |V|$
- Max edges?
  - $= O(|\boldsymbol{V}|^2) = O(\boldsymbol{n}^2)$
- So if a set of graphs (call it lang L) is decided by a TM where
  - # steps of the TM = polynomial in the # of vertices
  - Or # steps of the TM = polynomial in the # of edges
  - Then L is in $\mathbf{P}$

3 Problems in **P**

- A Graph Problem:

  $PATH = \{\langle G, s, t\rangle \mid G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$

- A Number Problem:

  $RELPRIME = \{\langle x, y\rangle \mid x \text{ and } y \text{ are relatively prime}\}$

- A CFL Problem:

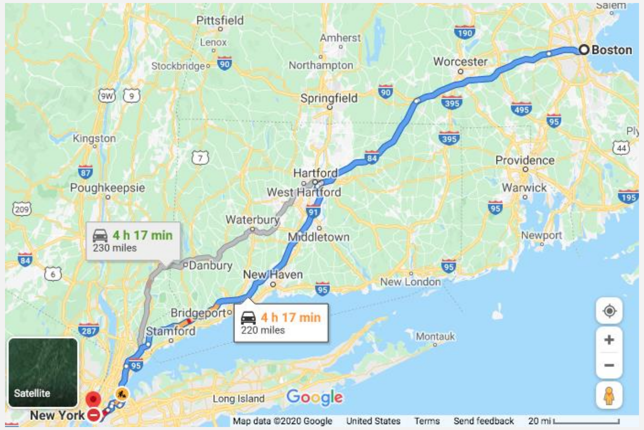
  Every context-free language is a member of P

P is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$\mathrm{P} = \bigcup_k \mathrm{TIME}(n^k).$$

A Graph Theorem: $PATH \in \mathrm{P}$

$$PATH = \{\langle G, s, t\rangle \mid G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$$



- To prove that a language is in **P** …
- … we must construct a <u>polynomial</u> time algorithm deciding the lang
- A <u>non-polynomial</u> (i.e., ”brute force”) algorithm:
  - check all possible paths, and see if any connect s to t
  - If n = # vertices, then # paths $\approx n^n$

Check-in Quiz 11/8

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