# CS622

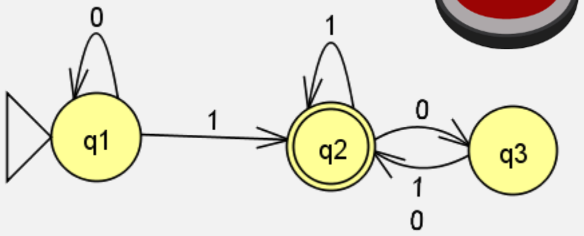
# Computing With DFAs, Formally

Monday, February 5, 2024

UMass Boston Computer Science

$\delta : Q \times \Sigma \longrightarrow Q$ is the *transition function*

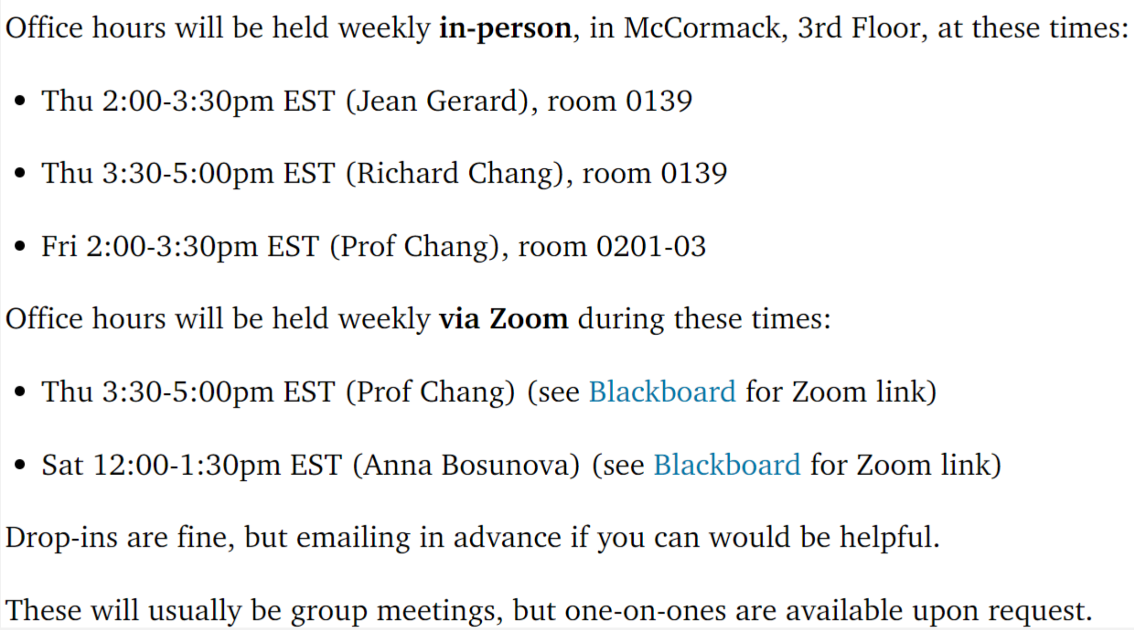
$\hat{\delta} : Q \times \Sigma^* \rightarrow Q$



# Announcements

- HW 1
  - Due: ~~Wed 2/7~~ Mon 2/12 12pm (noon)

- TAs and (new!) office hours



Office hours will be held weekly **in-person**, in McCormack, 3rd Floor, at these times:

- Thu 2:00-3:30pm EST (Jean Gerard), room 0139
- Thu 3:30-5:00pm EST (Richard Chang), room 0139
- Fri 2:00-3:30pm EST (Prof Chang), room 0201-03

Office hours will be held weekly **via Zoom** during these times:

- Thu 3:30-5:00pm EST (Prof Chang) (see Blackboard for Zoom link)
- Sat 12:00-1:30pm EST (Anna Bosunova) (see Blackboard for Zoom link)

Drop-ins are fine, but emailing in advance if you can would be helpful.

These will usually be group meetings, but one-on-ones are available upon request.

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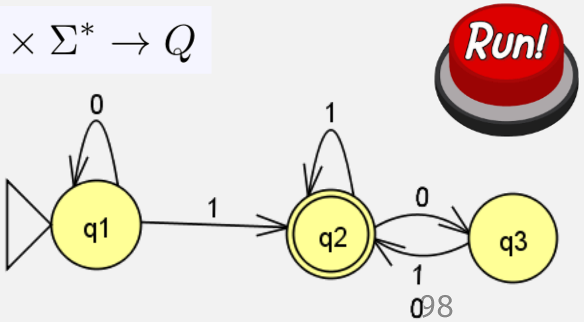
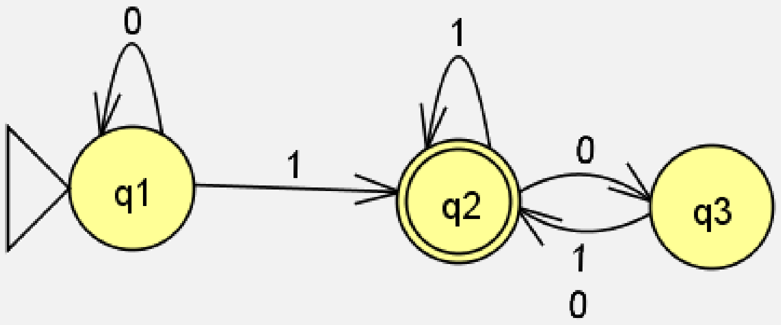


Previously

# Computation with DFAs (JFLAP demo)

- DFA:



- Input: “`1101`”

**HINT:** always work out concrete examples to understand how a machine works

*Previously*

# DFA Computation Rules

*Informally*

Given

- A **DFA** (~ a "Program")
- and <u>Input</u> **= string of chars**, e.g. "`1101`"

To **run** the automata / "program":

- <u>Start</u> in **"start state"**
- <u>Repeat</u>**:**
    - <u>Read</u> **1 char** from **input;**
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*Previously*

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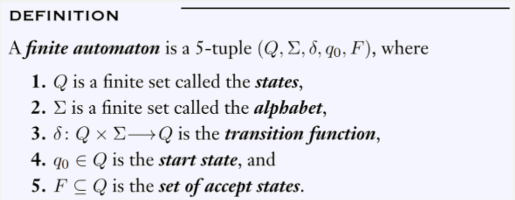

DEFINITION

A ***finite automaton*** is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. $Q$ is a finite set called the ***states***,
2. $\Sigma$ is a finite set called the ***alphabet***,
3. $\delta \colon Q \times \Sigma \longrightarrow Q$ is the ***transition function***,
4. $q_0 \in Q$ is the ***start state***, and
5. $F \subseteq Q$ is the ***set of accept states***.

## *Informally*

Given

- A **DFA** (~ a "Program")
- and <u>Input</u> = **string of chars**, e.g. `“1101”`

To **run** the automata / "program":

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## *Formally (i.e., mathematically)*

- $M =$
- $w =$

*Previously*

# DFA Computation Rules

## *Informally*

Given

- A **DFA** (~ a "Program")
- and **Input = string of chars**, e.g. "`1101`"

To **run** the automata / "program":

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- Repeat:
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## *Formally (i.e., mathematically)*

- $M = (Q, \Sigma, \delta, q_0, F)$
- $w = w_1 w_2 \cdots w_n$

A **run** is represented by **variables** $r_0$, ..., $r_n$ ,
the **sequence of states** in the **computation,** where:

- $r_0 = q_0$
- $M$ ***accepts*** $w$ if
  sequence of states $r_0, r_1, \ldots, r_n$ in $Q$ exists . . .
  with $r_n \in F$

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$\delta : Q \times \Sigma \longrightarrow Q$ is the ***transition function***

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- $w = \boxed{w_1} w_2 \cdots w_n$

A **run** is represented by **variables** $r_0, \ldots, r_n$ ,
the **<u>sequence of states</u>** in the **computation**, where:

- $\boxed{r_0} = q_0$
- $r_i =$

if $i$=1, $r_1 = \delta(r_0, w_1)$

if $i$=2, $r_2 = \delta(r_1, w_2)$

if $i$=3, $r_3 = \delta(r_2, w_3)$

- $M$ ***accepts*** $w$ if
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This is still a little "informal"

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$\delta : Q \times \Sigma \longrightarrow Q$ is the ***transition function***

# An Extended Transition Function

set of pairs

* = “0 or more”

Define **extended transition function:** $\hat{\delta} : Q \times \Sigma^* \to Q$

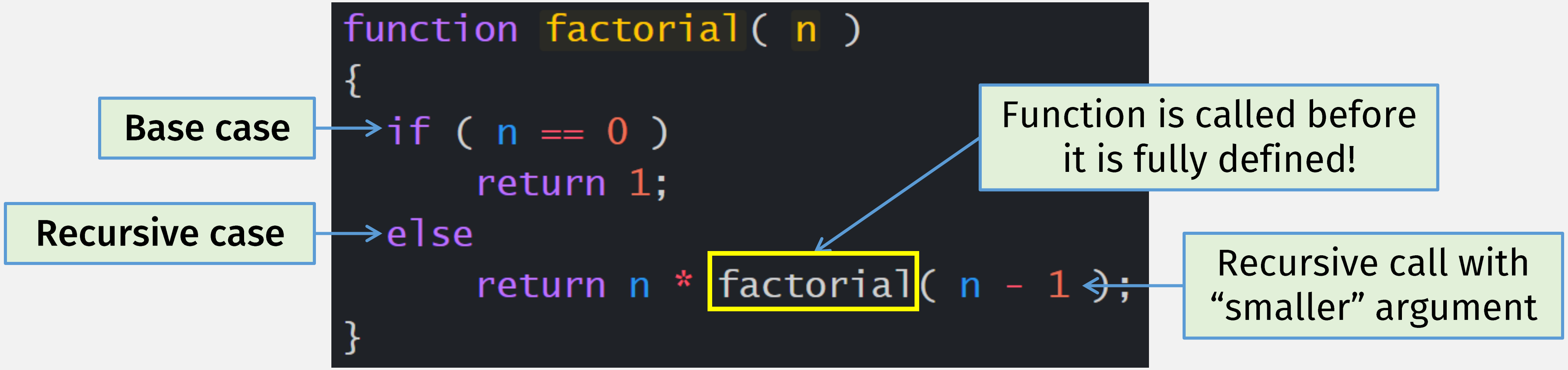
- <u>Domain</u>:
  - Input state $q \in Q$ (doesn’t have to be start state)
  - Input string $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$
- <u>Range</u>:
  - Output state (doesn’t have to be an accept state)

$\Sigma^*$ = set of all possible strings!

(Defined recursively)

- <u>Base case</u>: …

# *Interlude:* Recursive Definitions



Base case

Recursive case

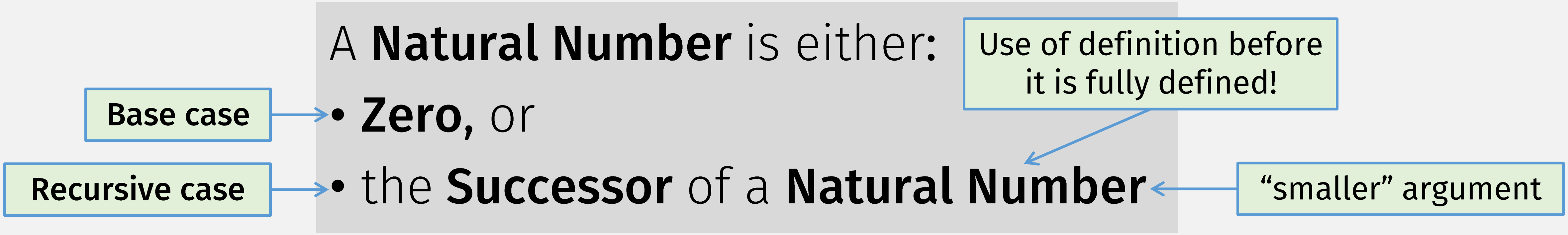
Function is called before it is fully defined!

Recursive call with “smaller” argument

```
function factorial( n )
{
  if ( n == 0 )
      return 1;
  else
      return n * factorial( n - 1 );
}
```

- Why is this allowed?
  - It’s a “**feature**” (i.e., an **axiom!**) of the **programming language**
- Why does this “work”? (Why doesn’t it loop forever?)
  - Because the **recursive call always has** a **“smaller” argument …**
  - … and so **eventually reaches** the **base case** and **stops**

# Recursive Definitions



A **Natural Number** is either:

- **Zero**, or
- the **Successor** of a **Natural Number**

**Base case**

**Recursive case**

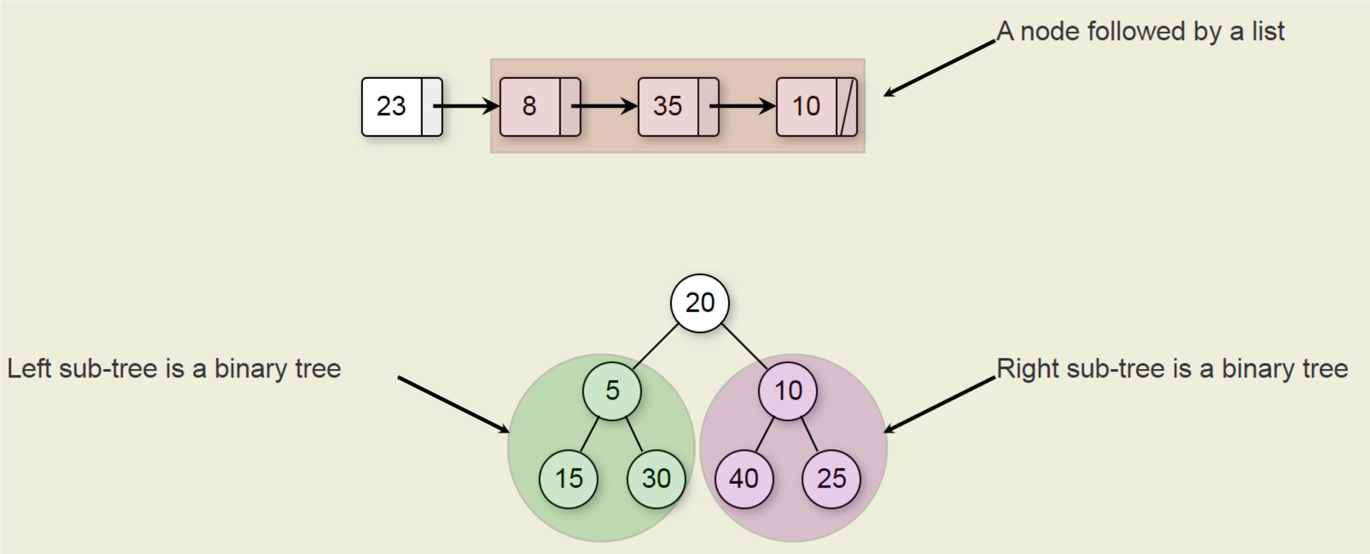
Use of definition before it is fully defined!

“smaller” argument

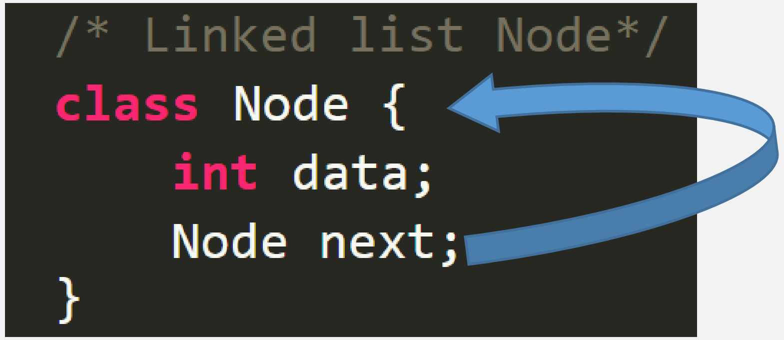
Examples

- **Zero**
- **Successor** of **Zero** ( = “one” )
- **Successor** of **Successor** of **Zero** ( = “two” )
- **Successor** of **Successor** of **Successor** of **Zero** ( = “three” ) …

# Recursive Definitions



Recursive definitions have:
- <u>base case</u> and
- <u>recursive case</u>
  (with a "smaller" object)



```
/* Linked list Node*/
class Node {
    int data;
    Node next;
}
```

This is a <u>recursive definition:</u>
**Node is used before it is fully defined** (but must be "smaller")

*Previously*

# Strings Are Defined Recursively

**Base case**

**Recursive case**

A **String** is either:

- the **empty string** (ε), or
- $xa$ **(non-empty string)** where
  - $x$ is a **string**
  - $a$ is a “**char**” in Σ

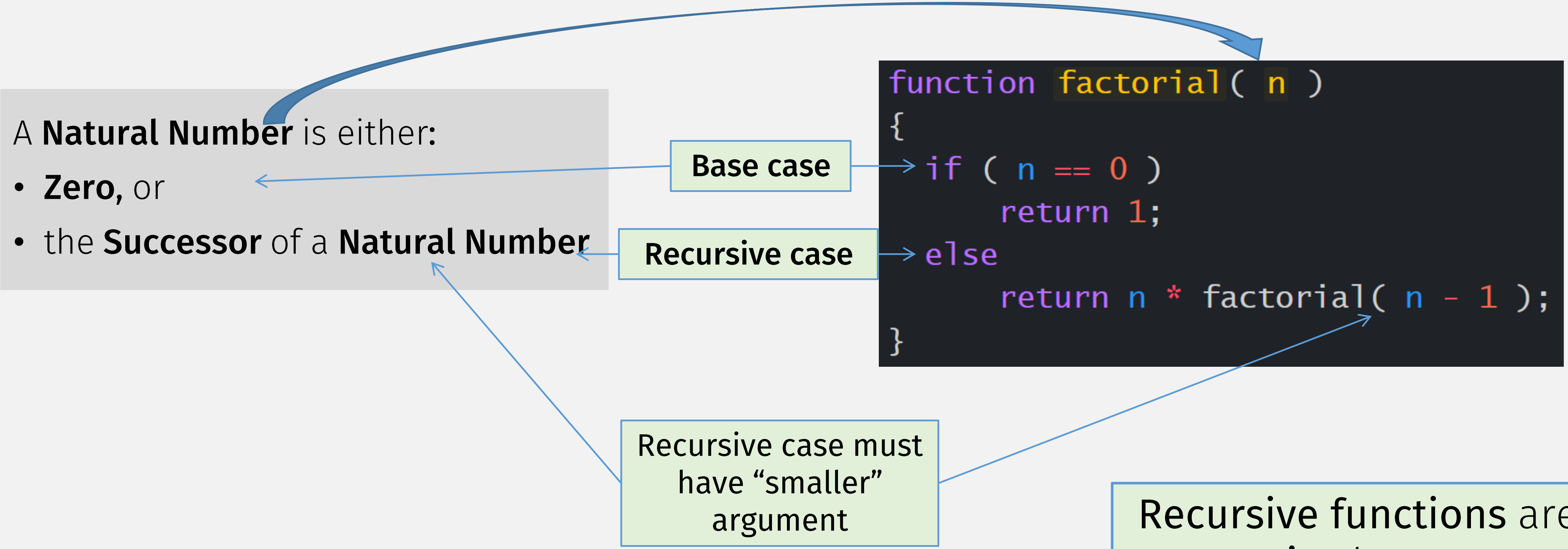
**“smaller” argument**

<u>Remember</u>: **all strings** are **formed with “chars”** from some **alphabet** set Σ

**$\Sigma^*$ = set of all possible strings!**

*Previously*

# Recursive Data ⇒ Recursive Functions



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```

Recursive case must have “smaller” argument

Recursive functions are **recursive** because … its **input data** is **recursively defined!**

# An Extended Transition Function

Define **extended transition function:** $\hat{\delta} : Q \times \Sigma^* \to Q$

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  - **Input string** $w = w_1 w_2 \cdots w_n$ where $w_i \in \Sigma$
- Range:
  - **Output state** (doesn't have to be an accept state)

Recursive Input Data
needs
Recursive Function

(Defined recursively)

- Base case $\hat{\delta}(q, \varepsilon) =$

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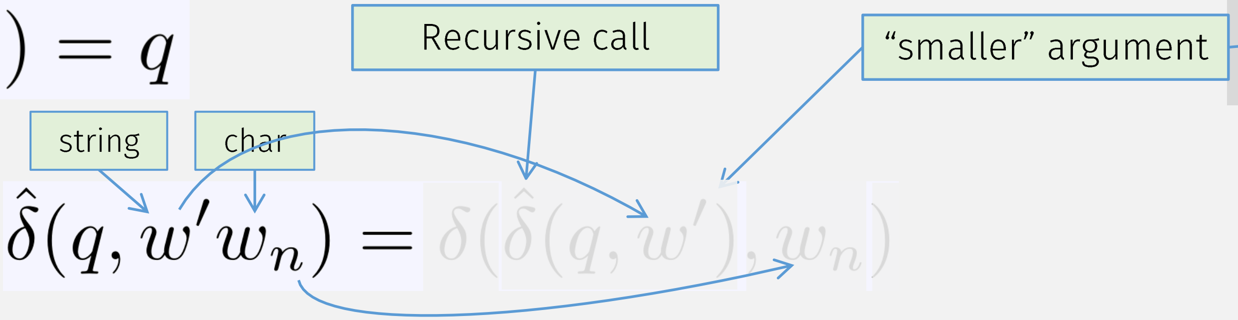

- Recursive Case $\hat{\delta}(q, w' w_n) = \delta(\hat{\delta}(q, w'), w_n)$

where $w' = w_1 \cdots w_{n-1}$

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*Previously*

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- $M$ ***accepts*** $w$ if $\hat{\delta}(q_0, w) \in F$

sequence of states $r_0, r_1, \ldots, r_n$ in $Q$ exists …

with $r_n \in F$

# Definition of Accepting Computations

An **accepting computation**, for DFA $M = (Q, \Sigma, \delta, q_0, F)$ and **string** $w$:

1. **starts** in the **start state** $q_0$

2. **goes through** a **valid sequence** of **states according to** $\delta$

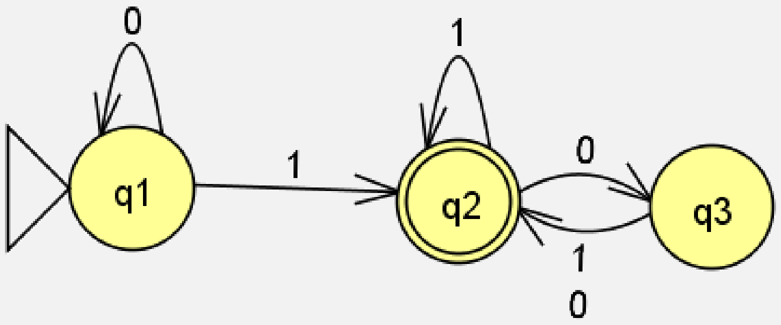
3. **ends** in an **accept state**

All 3 must be true for a computation to be an **accepting computation!**

$M$ ***accepts*** $w$ if $\hat{\delta}(q_0, w) \in F$

# Accepting Computation or Not?

**DFA:**



- $\hat{\delta}$ (q1, **`1101`**)
  - Yes
- $\hat{\delta}$ (q1, **`110`**)
  - **No** (doesn't end in accept state)
- $\hat{\delta}$ (q2, **`101`**)
  - **No** (doesn't start in start state)

# Alphabets, Strings, Languages

Alphabet specifies “all possible strings”

- An **alphabet** is a non-empty finite set of **symbols**

(impossible to have strings with non-alphabet chars)

$$\Sigma_1 = \{\mathtt{0},\mathtt{1}\}$$

$$\Sigma_2 = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}, \mathtt{g}, \mathtt{h}, \mathtt{i}, \mathtt{j}, \mathtt{k}, \mathtt{l}, \mathtt{m}, \mathtt{n}, \mathtt{o}, \mathtt{p}, \mathtt{q}, \mathtt{r}, \mathtt{s}, \mathtt{t}, \mathtt{u}, \mathtt{v}, \mathtt{w}, \mathtt{x}, \mathtt{y}, \mathtt{z}\}$$

- A **string** is a finite sequence of **symbols** from an **alphabet**

`01001` `abracadabra` $\varepsilon$ ← Empty string (length 0)

- A **language** is a set of **strings**

Languages can be infinite

$$A = \{\mathtt{good}, \mathtt{bad}\}$$

$$\emptyset \quad \{\ \}$$

Empty set is a language

$$A = \{w \mid w \text{ contains at least one } \mathtt{1} \text{ and an even number of } \mathtt{0}\text{s follow the last } \mathtt{1}\}$$

“the set of all …”

“such that …”

# Computation and Languages

- The **language** of a **machine** is the <u>set</u> **of all strings** that it **accepts**

- E.g., A **DFA** $M$ ***accepts*** $w$ if $\hat{\delta}(q_0, w) \in F$

- **Language** of $M = L(M) = \{w \mid M \text{ accepts } w\}$

"the set of all …"

"such that …"

# Machine and Language Terminology

DFA $M$ ***accepts*** $w$

string

$M$ ***recognizes language*** $A$

Set of strings

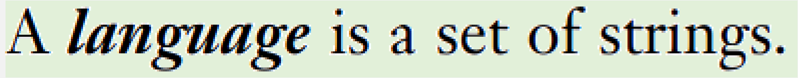
if $A = \{w \mid M \text{ accepts } w\}$

# Computation and Classes of Languages

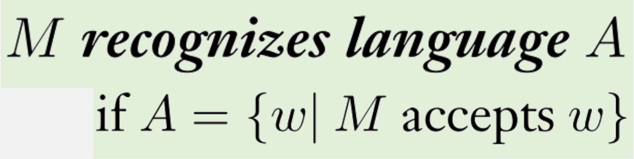
- The **language** of a **machine** = **set** of **all strings** that it **accepts**
  - E.g., every **DFA** is **associated with** a **language**
- A **computation model** = **set** of **machines** it **defines**
  - E.g., **all possible DFAs** are a **computation model**
- Thus: a **computation model** = **set** of **languages**

# Regular Languages: Definition

If a **deterministic finite automata** (**DFA**) recognizes a language, then **that language** is called a **regular language**.



A ***language*** is a set of strings.



$M$ ***recognizes language*** $A$ if $A = \{w|\ M \text{ accepts } w\}$

# A Language, Regular or Not?

- If given: a DFA $M$
  - We know: $L(M)$, the language recognized by $M$, is a **regular language**

If a DFA recognizes a language,
then that language is called a **regular language**.

(modus ponens)

- If given: a Language $A$
  - Is $A$ a regular language?
    - Not necessarily!
  - How do we determine, i.e., *prove*, that $A$ is a regular language?

# An Inference Rule: Modus Ponens

**Premises**

- If $P$ then $Q$
- $P$ is true

**Conclusion**

- $Q$ must also be true

Example **Premises**

- If **there is an DFA recognizing language $A$,** then $A$ **is a regular language**
- There is an DFA $M$ where $L(M) = A$

**Conclusion**

- $A$ is a regular language!

# A Language, Regular or Not?

- If given: a DFA $M$
  - We know: $L(M)$, the language recognized by $M$, is a regular language

If a DFA <u>recognizes</u> a language,
then that language is called a **regular language**.

- If given: a Language $A$
  - Is $A$ a regular language?
    - Not necessarily!
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Prove there is a DFA recognizing $A$!

**HINT:** always work out concrete examples to understand a language

# Language: strs with odd # **1**s

| Example | In the language? |
|---|---|
| 1 | Yes |
| 0 | No |
| 01 | Yes |
| 11 | No |
| 1101 | Yes |
| ε | no |

$\Sigma = \{0,1\}$

If a DFA recognizes a language,
then that language is called a **regular language**.

How to prove the language is regular?

Prove there's a DFA recognizing it!

# Designing Finite Automata: Tips

- Input is read only once, one char at a time

- Must decide accept/reject after that

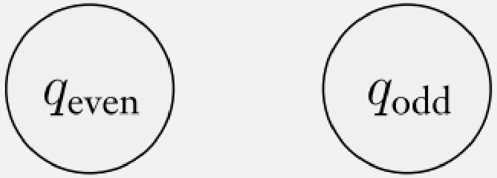
- States = the machine’s **memory**!
    - # states must be decided in advance
    - Think about what information must be remembered.

- Every state/symbol pair must have a transition (for DFAs)

- Come up with examples!

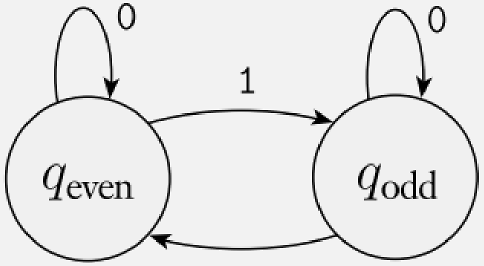
# Design a DFA: accept strs with odd # **1**s

- States:
  - 2 states:
    - seen even **1**s so far
    - seen odds **1**s so far

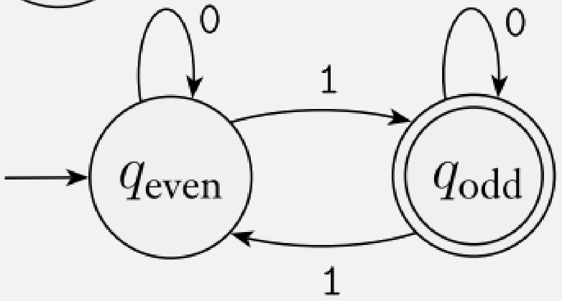


- Alphabet: `0` and `1`

- Transitions:



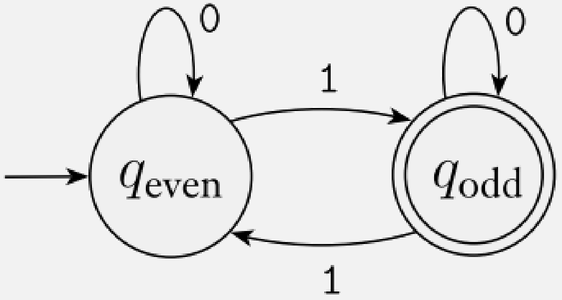
- Start / Accept states:



# “Prove” that DFA recognizes a language

| Example | In the language? |
|---|---|
| 1 | Yes |
| 0 | No |
| 01 | Yes |
| 11 | No |
| 1101 | Yes |
| ε | no |

$\Sigma = \{0,1\}$



In this class, **a table like this is sufficient to “prove” that a DFA recognizes a language**

# Submit 2/5 in-class work to gradescope