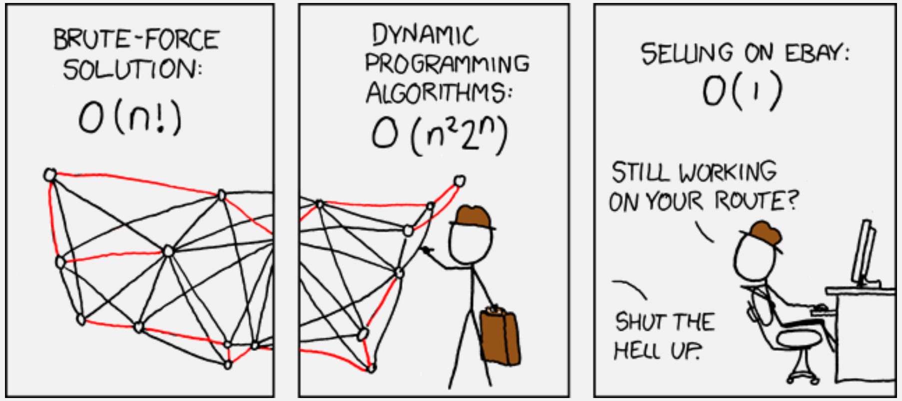
UMB CS622

P (and NP?)

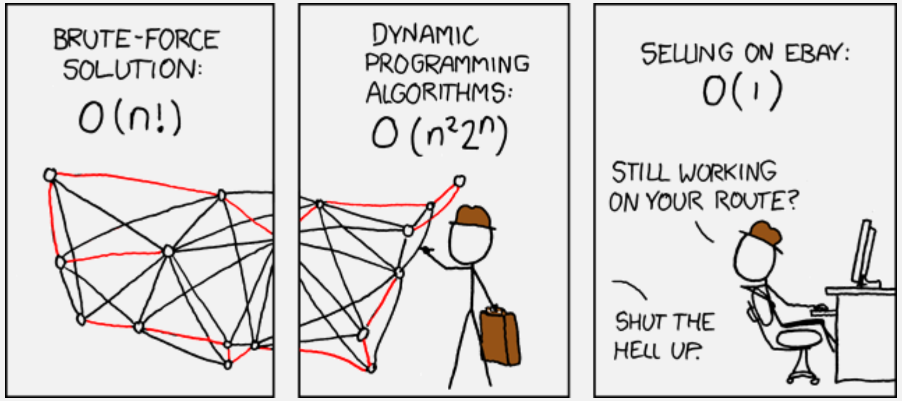
Monday, May 6, 2024



Announcements

- HW 11
  - Due Wed 5/8 12pm noon

- HW 12
  - Release Wed 5/8 12pm noon
  - Due Wed 5/15 12pm noon (no late days, no exceptions)



3 Problems in **P**

☑ • A <u>Graph</u> Problem:

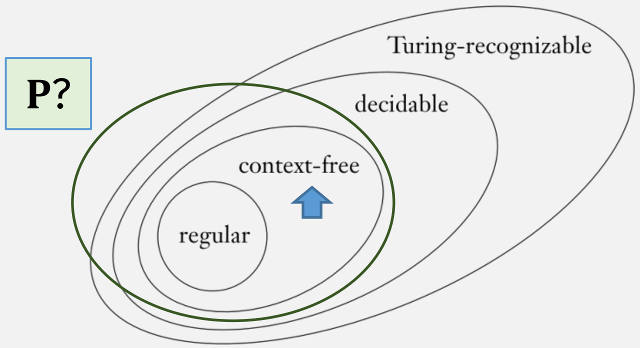
$PATH = \{\langle G, s, t\rangle \mid G$ is a directed graph that has a directed path from s to $t\}$

☑ • A <u>Number</u> Problem:

$RELPRIME = \{\langle x, y\rangle \mid x$ and y are relatively prime$\}$

• A <u>CFL</u> Problem:

Every context-free language is a member of P



IF-THEN Statement to Prove:

IF a language L is a CFL,
THEN L is in **P**

Review: A Decider for Any CFL

Given any CFL L, with CFG G, the following decider M_G decides L:

$M_G =$ "On input w:
1. Run TM S on input $\langle G, w \rangle$.
2. If this machine accepts, *accept*; if it rejects, *reject*."

S is a decider for: $A_{\text{CFG}} = \{\langle G, w \rangle | \ G \text{ is a CFG that generates string } w\}$

$S =$ "On input $\langle G, w \rangle$, where G is a CFG and w is a string:
1. Convert G to an equivalent grammar in Chomsky normal form.
2. List all derivations with $2n - 1$ steps, where n is the length of w; except if $n = 0$, then instead list all derivations with one step.
3. If any of these derivations generate w, *accept*; if not, *reject*."

M_G is a decider, bc S is a decider

M_G accepts all $w \in L$, for any CFL L (with CFL G)

Therefore, every CFL is decidable

But, is every CFL decidable in poly time?

A Decider for Any CFL: Running Time

Given any CFL L, with CFG G, the following decider M_G decides L:

$M_G =$ "On input w:
1. Run TM S on input $\langle G, w\rangle$.
2. If this machine accepts, *accept*; if it rejects, *reject*."

S is a decider for: $A_{\text{CFG}} = \{\langle G, w\rangle \mid G \text{ is a CFG that generates string } w\}$

$A \to 0A1$
$A \to B$
$B \to \#$

How many different possibilities at *each* derivation step?

$A \Rightarrow 0A1 \Rightarrow \quad \ldots$

$S =$ "On input $\langle G, w\rangle$, where G is a CFG and w is a string:
1. Convert G to an equivalent grammar in Chomsky normal form.
2. List all derivations with $2n-1$ steps, where n is the length of w; except if $n = 0$, then instead list all derivations with one step.
3. If any of these derivations generate w, *accept*; if not, *reject*."

Worst case:
$|R|^{2n-1}$ steps $= O(2^n)$
(R = set of rules)

This algorithm runs in exponential time

A CFL Theorem: Every context-free language is a member of P

- Given a CFL, we must construct a decider for it …
- … that runs in polynomial time

Dynamic Programming

- **Keep track of partial solutions,** and **re-use them**
  - Start with smallest and build up
- **For CFG problem,** instead **of re-generating entire string …**
  - … keep track of <u>substrings</u> generated by each variable

$S =$ "On input $\langle G, w \rangle$, where G is a CFG and w is a string:

1. Convert G to an equivalent grammar in Chomsky normal form.
2. List all derivations with $2n - 1$ steps, where n is the length of w; except if $n = 0$, then instead list all derivations with one step.
3. If any of these derivations generate w, *accept*; if not, *reject*."

This <u>duplicates a lot of work</u> because many strings might have the <u>same beginning derivation steps</u>

CFL Dynamic Programming Example

- Chomsky Grammar G:
  - S → AB | BC
  - A → BA | a
  - B → CC | b
  - C → AB | a
- Example string: **baaba**
- Store every partial string and their generating variables in a table

Substring end char

Substring start char

	b	**a**	**a**	**b**	**a**
b					
a					
a					
b					
a					

CFL Dynamic Programming Example

- Chomsky Grammar G:
  - S → AB | BC
  - A → BA | a
  - B → CC | b
  - C → AB | a
- Example string: **baaba**
- Store every partial string and their generating variables in a table

Substring <u>end</u> char

Substring <u>start</u> char

	b	**a**	**a**	**b**	**a**
b	vars generating "b"	vars for "ba"	vars for "baa"	…	
a		vars for "a"	vars for "aa"	vars for "aab"	
a			…		
b					
a					

CFL Dynamic Programming Example

- Chomsky Grammar G:
  - S → AB | BC
  - A → BA | a
  - B → CC | b
  - C → AB | a
- Example string: **baaba**
- Store every partial string and their generating variables in a table

Algo:
- **For** each single char c and var A:
  - **If** A → c is a rule, add A to table

Substring end char

Substring start char

	b	**a**	**a**	**b**	**a**
b	vars generating "b"	vars for "ba"	vars for "baa"	…	
a		vars for "a"	vars for "aa"	vars for "aab"	
a			…		
b					
a					

CFL Dynamic Programming Example

- Chomsky Grammar G:
  - S → AB | BC
  - A → BA | a
  - B → CC | b
  - C → AB | a
- Example string: **baaba**
- Store every partial string and their generating variables in a table

Algo:
- For each single char c and var A:
  - If A → c is a rule, add A to table

Substring end char

Substring start char

	b	**a**	**a**	**b**	**a**
b	B				
a		A,C			
a			A,C		
b				B	
a					A,C

CFL Dynamic Programming Example

- Chomsky Grammar G:
  - $S \rightarrow AB \mid BC$
  - $A \rightarrow BA \mid a$
  - $B \rightarrow CC \mid b$
  - $C \rightarrow AB \mid a$
- Example string: **baaba**
- Store every partial string and their generating variables in a table

Algo:
- For each single char c and var A:
  - If $A \rightarrow c$ is a rule, add A to table
- For each substring s (len > 1):
  - For each split of substring s into x,y:
    - For each rule of shape $A \rightarrow BC$:
      - Use table to check if B generates x and C generates y

Substring end char

Substring start char

	b	a	a	b	a
b	B				
a		A,C			
a			A,C		
b				B	
a					A,C

CFL Dynamic Programming Example

- Chomsky Grammar G:
  - S → AB | BC
  - A → BA | a
  - B → CC | b
  - C → AB | a
- Example string: **baaba**
- Store every partial string and their ge[nerating]

Algo:
- For each single char c and var A:
  - If A → c is a rule, add A to table
- For each substring s:
  - For each split of substring s into x,y:
    - For each rule of shape A → BC:
      - Use table to check if B

Substring end char

Substring start char

	b	a	a
b	B		
a		A,C	
a			A,C
b			
a			

For substring "**ba**", split into "**b**" and "**a**":
- For rule S → AB
  - Does A generate "**b**" and B generate "**a**"?
  - NO
- For rule S → BC
  - Does B generate "**b**" and C generate "**a**"?
  - **YES**
- For rule A → BA
  - Does B generate "**b**" and A generate "**a**"?
  - **YES**
- For rule B → CC
  - Does C generate "**b**" and C generate "**a**"?
  - NO
- For rule C → AB
  - Does A generate "**b**" and B generate "**a**"?
  - NO

CFL Dynamic Programming Example

- Chomsky Grammar G:
  - S → AB | BC
  - A → BA | a
  - B → CC | b
  - C → AB | a
- Example string: **baaba**
- Store every partial string and their generating

Algo:
- For each single char c and var A:
  - If A → c is a rule, add A to table
- For each substring s:
  - For each split of substring s into x,y:
    - For each rule of shape A → BC:
      - Use table to check if B

For substring "**ba**", split into "**b**" and "**a**":
- For rule S → AB
  - Does A generate "**b**" and B generate "**a**"?
  - NO
- For rule S → BC
  - Does B generate "**b**" and C generate "**a**"?
  - **YES**
- For rule A → BA
  - Does B generate "**b**" and A generate "**a**"?
  - **YES**
- For rule B → CC
  - Does C generate "**b**" and C generate "**a**"?
  - NO
- For rule C → AB
  - Does A generate "**b**" and B generate "**a**"?
  - NO

Substring end char

Substring start char

	b	**a**	**a**
b	B	S,A	
a		A,C	
a			A,C
b			
a			

CFL Dynamic Programming Example

- Chomsky Grammar G:
  - … C
  - …
  - B → CC | b
  - C → AB | a
- …ing: **baaba**
- … partial string and their ge… variables in a table

For each:
- char
- var

For each:
- substring
- split of substring
- rule

Algo:

For each: char, var …

- For each single char c and var A:
  - If A → c is a rule, add A to table
- For each substring … For each: substring, split, rule …
  - For each split of substring s into x,y:
    - For each rule of shape A → BC:
      - Use table to check if B generates x and C generates y

Substring end char

Substring start char

If S is here, accept

	b	a	a	b	a
b	B	S,A			S,A,C
a		A,C	B	B	S,A,C
a			A,C	S,C	B
b				B	S,A
a					A,C

A CFG Theorem: Every context-free language is a member of P

D = "On input $w = w_1 \cdots w_n$:

1. For $w = \varepsilon$, if $S \to \varepsilon$ is a rule, *accept*; else, *reject*. [[$w = \varepsilon$ case]]
2. For $i = 1$ to n: [[examine each substring of length 1]]
3. For each variable A:
4. Test whether $A \to \mathtt{b}$ is a rule, where $\mathtt{b} = w_i$.
5. If so, place A in *table*(i, i).
6. For $l = 2$ to n: [[l is the length of the substring]]
7. For $i = 1$ to $n - l + 1$: [[... substring]]
8. Let $j = i + l - 1$. [[j is the end position of the substring]]
9. For $k = i$ to $j - 1$:
10. For each rule $A \to BC$:
11. If *table*(i, k) contains B and *table*$(k + 1, j)$ contains C, put A in *table*(i, j).
12. If S is in *table*$(1, n)$, *accept*; else, *reject*."

For each:
- char
- var

$O(\boldsymbol{n})$ chars

#vars = constant = $O(\mathbf{1})$

$O(\mathbf{1}) * O(\boldsymbol{n}) = O(\boldsymbol{n})$

For each:
- substring
- split of substring
- rule

$O(\boldsymbol{n})$ diff lengths

$O(\boldsymbol{n})$ strings of each length

$O(\boldsymbol{n})$ ways to split a string into two pieces

#vars = constant = $O(\mathbf{1})$

$O(\mathbf{1}) * O(\boldsymbol{n}) * O(\boldsymbol{n}) * O(\boldsymbol{n}) = O(\boldsymbol{n}^3)$

Total: $\boxed{O(\boldsymbol{n}^3)}$

(This is also known as the Earley parsing algorithm)

Summary: 3 Problems in $\mathbf{P}$

☑ • A Graph Problem:

$PATH = \{\langle G, s, t\rangle |\ G$ is a directed graph that has a directed path from s to $t\}$

☑ • A Number Problem:

$RELPRIME = \{\langle x, y\rangle |\ x$ and y are relatively prime$\}$

☑ • A CFL Problem:

Every context-free language is a member of P

NP

Who doesn't like niche NP jokes?

Last Time: Poly Time Complexity Class (**P**)

P is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$\mathrm{P} = \bigcup_k \mathrm{TIME}(n^k).$$

- Corresponds to "realistically" solvable problems:
  - Problems in **P**
    - = "solvable" or "tractable"
  - Problems outside **P**
    - = "unsolvable" or "intractable"

Last Time: 3 Problems in **P**

“search” problem

- A Graph Problem:

  $PATH = \{\langle G, s, t\rangle |\ G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$

- A Number Problem:

  $RELPRIME = \{\langle x, y\rangle |\ x \text{ and } y \text{ are relatively prime}\}$

- A CFL Problem:

  Every context-free language is a member of P

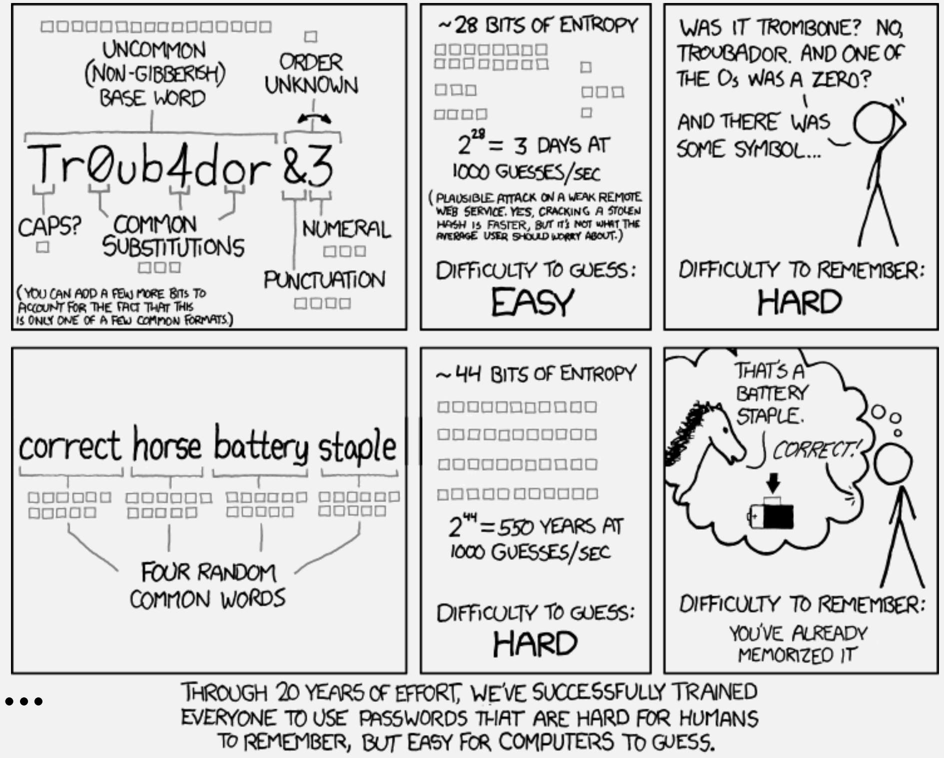
Search vs Verification

- **<u>Search</u>** problems are often **unsolvable**
- But, **<u>verification</u>** of a search result is usually **solvable**

Examples

- Factoring
  - **Unsolvable**: <u>Find</u> factors of 8633
    - Must “try all” possibilities
  - **Solvable**: <u>Verify</u> 89 and 97 are factors of 8633
    - Just do multiplication

- Passwords
  - **Unsolvable**: <u>Find</u> my `umb.edu` password
  - **Solvable** : <u>Verify</u> whether my `umb.edu` password is …
    - “`correct horse battery staple`”



The *PATH* Problem

$PATH = \{\langle G, s, t \rangle \mid G$ is a directed graph that has a directed path from s to $t\}$

- It's a **search** problem:
  - Exponential time (brute force) algorithm (n^n):
    - Check all n^n possible paths and see if any connects s and t
  - Polynomial time algorithm:
    - Do a breadth-first search (roughly), marking "seen" nodes as we go ($\boldsymbol{n}$ = # nodes)

PROOF A polynomial time algorithm M for *PATH* operates as follows.

M = "On input $\langle G, s, t \rangle$, where G is a directed graph with nodes s and t:

1. Place a mark on node s.
2. Repeat the following until no additional nodes are marked:
3. Scan all the edges of G. If an edge (a, b) is found going from a marked node a to an unmarked node b, mark node b.
4. If t is marked, *accept*. Otherwise, *reject*."

$O(\boldsymbol{n}^3)$

Verifying a *PATH*

$PATH = \{\langle G, s, t\rangle | \ G \text{ is a directed graph that has a directed path from } s \text{ to } t\}$

The **verification** problem:

- Given some path p in G, check that it is a path from s to t
- Let $\boldsymbol{m}$ = length of longest possible path = # ed

NOTE: extra argument p,
"**Verifying**" an answer requires
having a potential answer to check!

Verifier V = On input <G, s, t, p>, where p is some set of edges:

1. Check some edge in p has "from" node s; mark and set it as "current" edge
   - Max steps = $O(\boldsymbol{m})$
2. **Loop**: While there remains unmarked edges in p:
   1. Find the "next" edge in p, whose "from" node is the "to" node of "current" edge
   2. If found, then mark that edge and set it as "current", else reject
   - Each loop iteration: $O(\boldsymbol{m})$
   - # loops: $O(\boldsymbol{m})$
   - Total looping time = $O(\boldsymbol{m}^2)$
3. Check "current" edge has "to" node t; if yes accept, else reject

☑

- Total time = $O(\boldsymbol{m}) + O(\boldsymbol{m}^2) = \boxed{O(\boldsymbol{m}^2)}$ = polynomial in m

PATH can be **verified**
in polynomial time

Verifiers, Formally

$PATH = \{\langle G, s, t\rangle \mid G$ is a directed graph that has a $\boxed{\text{directed path}}$ from s to $t\}$

A possible

Decider …

… with extra argument:
can be any string that helps
to find a result in poly time
(is often just a potential
result itself)

A ***verifier*** for a language A is an algorithm V, where

$$A = \{w \mid V \text{ accepts } \langle w, c\rangle \text{ for some string } c\}.$$

certificate, or ***proof***

We measure the time of a verifier only in terms of the length of w, so a ***polynomial time verifier*** runs in polynomial time in the length of w. A language A is ***polynomially verifiable*** if it has a polynomial time verifier.

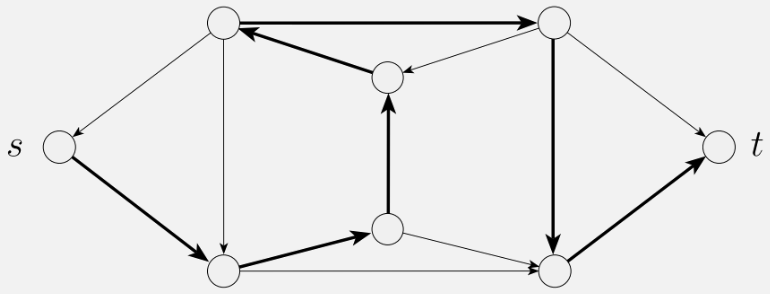
- NOTE: a certificate c must be at most length n^k, where n = length of w
  - Why? Because it takes time n^k to read it

So *PATH* is polynomially verifiable

The *HAMPATH* Problem

$$\textit{HAMPATH} = \{\langle G, s, t\rangle \mid G \text{ is a directed graph with a Hamiltonian path from } s \text{ to } t\}$$

- A **Hamiltonian path** goes through every node in the graph



- The **Search** problem:
  - Exponential time (brute force) algorithm:
    - Check all possible paths and see if any connect s and t using all nodes
  - Polynomial time algorithm: **???**
    - We don't know if there is one!!!
- The **Verification** problem:
  - Still $O(m^2)$! (same verifier for *PATH*)
  - *HAMPATH* is polynomially verifiable, but not polynomially decidable

The class **NP**

DEFINITION

NP is the class of languages that have polynomial time $\boxed{\text{verifiers.}}$

- *PATH* is in **NP**, and **P**
- *HAMPATH* is in **NP**, but it's <u>unknown</u> whether it's in **P**

NP = Nondeterministic polynomial time

NP is the class of languages that have polynomial time verifiers.

THEOREM

A language is in NP iff it is decided by some nondeterministic polynomial time Turing machine.

$\Rightarrow$ If a language is in **NP**, then it has a non-deterministic poly time decider

- We know: If a lang L is in **NP**, then it has a poly time verifier V
- Need to: create NTM deciding L:

On input w =

- Nondeterministically run V with w and all possible poly length certificates c

NOTE: a verifier **cert** is usually a potential "answer", but does not have to be (like here)

$\Leftarrow$ If a language has a non-deterministic poly time decider, then it is in **NP**

- We know: L has NTM decider N,
- Need to: show L is in **NP**, i.e., create polytime verifier V:

On input $<w, c>$ = Potentially exponential slowdown?

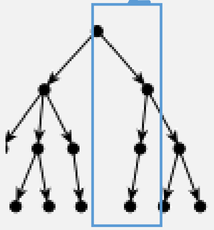
But which path to take?

- Convert N to deterministic TM, and run it on w, but take only one computation path
- Let certificate c dictate which computation path to follow

Deterministic (verifier) TMs cannot "call" non-deterministic TMs

Converting NTM to deterministic TM is exponentially slower!

Certificate c specifies a path



NP

$$\textbf{NTIME}(\boldsymbol{t(n)}) = \{L \mid L \text{ is a language decided by an } O(t(n)) \text{ time nondeterministic Turing machine}\}.$$

$$\mathrm{NP} = \bigcup_k \mathrm{NTIME}(n^k)$$

NP = Nondeterministic polynomial time

NP vs **P**

P is the class of languages that are decidable in polynomial time on a deterministic single-tape Turing machine. In other words,

$$\mathrm{P} = \bigcup_k \mathrm{TIME}(n^k).$$

P = <u>Deterministic</u> polynomial time

$$\textbf{NTIME}(\boldsymbol{t(n)}) = \{L \mid L \text{ is a language decided by an } O(t(n)) \text{ time nondeterministic Turing machine}\}.$$

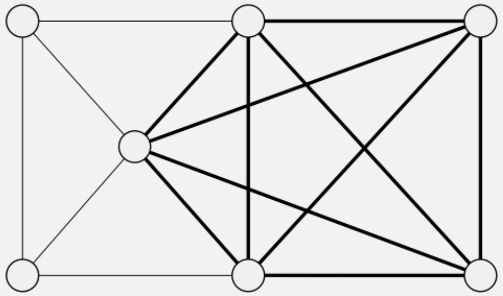
$$\mathrm{NP} = \bigcup_k \mathrm{NTIME}(n^k)$$

Also, **NP** = <u>Deterministic</u> polynomial time **verification**

NP = <u>Nondeterministic</u> polynomial time

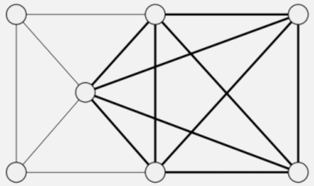
More **NP** Problems

- $CLIQUE = \{\langle G, k\rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$
  - A clique is a subgraph where every two nodes are connected
  - A k-clique contains k nodes



- $$SUBSET\text{-}SUM = \{\langle S, t\rangle \mid S = \{x_1, \ldots, x_k\}, \text{ and for some } \{y_1, \ldots, y_l\} \subseteq \{x_1, \ldots, x_k\}, \text{ we have } \Sigma y_i = t\}$$

Theorem: *CLIQUE* is in NP



$CLIQUE = \{\langle G, k\rangle |\ G \text{ is an undirected graph with a } k\text{-clique}\}$

PROOF IDEA The clique is the certificate.

Let $\boldsymbol{n}$ = # nodes in G

c is at most $\boldsymbol{n}$

PROOF The following is a $\boxed{\text{verifier } V}$ for *CLIQUE*.

V = "On input $\langle\langle G, k\rangle, c\rangle$:

1. Test whether c is a subgraph with k nodes in G.
2. Test whether G contains all edges connecting nodes in c.
3. If both pass, *accept*; otherwise, *reject*."

`For each:` node in c,
check whether it's in G
$O(\boldsymbol{n})$

`For each:` pair of nodes in c,
check whether there's an edge in G:
$O(\boldsymbol{n}^2)$

A ***verifier*** for a language A is an algorithm V, where

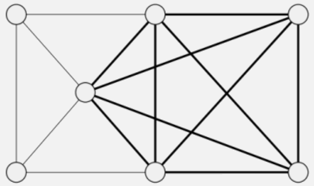
$$A = \{w |\ V \text{ accepts } \langle w, c\rangle \text{ for some string } c\}.$$

We measure the time of a verifier only in terms of the length of w, so a ***polynomial time verifier*** runs in polynomial time in the length of w. A language A is ***polynomially verifiable*** if it has a polynomial time verifier.

How to prove a language is in **NP**:
Proof technique #1: **create a verifier**

NP is the class of languages that have polynomial time verifiers.

Proof 2: *CLIQUE* is in NP



$CLIQUE = \{\langle G, k\rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$

$N =$ "On input $\langle G, k\rangle$, where G is a graph:

1. Nondeterministically select a subset c of k nodes of G.
2. Test whether G contains all edges connecting nodes in c.
3. If yes, *accept*; otherwise, *reject*."

"try all subgraphs"

Checking whether a subgraph is clique: $O(\boldsymbol{n^2})$

To prove a lang L is in **NP**, create either a:

1. **Deterministic** poly time **verifier**
2. **Nondeterministic** poly time **decider**

How to prove a language is in **NP**:
Proof technique #2: **create an NTM**

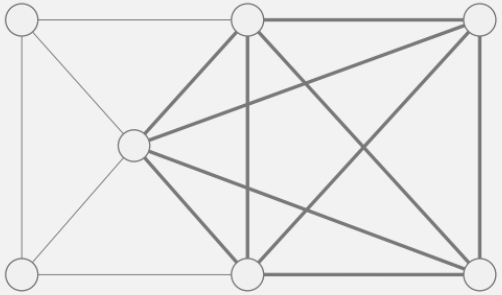
Don't forget to count the steps

THEOREM

A language is in NP iff it is decided by some nondeterministic polynomial time Turing machine.

More **NP** Problems

- $\mathit{CLIQUE} = \{\langle G, k\rangle \mid G \text{ is an undirected graph with a } k\text{-clique}\}$
  - A clique is a subgraph where every two nodes are connected
  - A k-clique contains k nodes



set sum

- $\mathit{SUBSET\text{-}SUM} = \{\langle S, t\rangle \mid S = \{x_1, \ldots, x_k\}, \text{ and for some}$

  subset $\{y_1, \ldots, y_l\} \subseteq \{x_1, \ldots, x_k\}, \text{ we have } \Sigma y_i = t\}$ sum
  - Some subset of a set of numbers S must sum to some total t
  - e.g., $\langle\{\boxed{4}, 11, 16, \boxed{21}, 27\}, 25\rangle \in \mathit{SUBSET\text{-}SUM}$

Theorem: *SUBSET-SUM* is in NP

$$SUBSET\text{-}SUM = \{\langle S, t\rangle \mid S = \{x_1, \ldots, x_k\}, \text{ and for some } \{y_1, \ldots, y_l\} \subseteq \{x_1, \ldots, x_k\}, \text{ we have } \Sigma y_i = t\}$$

PROOF IDEA The subset is the certificate.

To prove a lang is in **NP**, create either:
1. **Deterministic** poly time **verifier**
2. **Nondeterministic** poly time **decider**

PROOF The following is a verifier V for *SUBSET-SUM*.

V = "On input $\langle\langle S, t\rangle, c\rangle$:

1. Test whether c is a collection of numbers that sum to t.
2. Test whether S contains all the numbers in c.
3. If both pass, *accept*; otherwise, *reject*."

Don't forget to compute run time!
Does this run in poly time?

Proof 2: *SUBSET-SUM* is in NP

$$SUBSET\text{-}SUM = \{\langle S, t\rangle \mid S = \{x_1, \ldots, x_k\}, \text{ and for some } \{y_1, \ldots, y_l\} \subseteq \{x_1, \ldots, x_k\}, \text{ we have } \Sigma y_i = t\}$$

To prove a lang is in **NP**, create either:

1. **Deterministic** poly time **verifier**
2. **Nondeterministic** poly time **decider**

Don't forget to compute run time!
Does this run in poly time?

ALTERNATIVE PROOF We can also prove this theorem by giving a nondeterministic polynomial time Turing machine for *SUBSET-SUM* as follows.

N = "On input $\langle S, t\rangle$:

1. Nondeterministically select a subset c of the numbers in S.
2. Test whether c is a collection of numbers that sum to t.
3. If the test passes, *accept*; otherwise, *reject*."

Nondeterministically runs the verifier on each possible subset in parallel

$$COMPOSITES = \{x|\, x = pq,\ \text{for integers } p, q > 1\}$$

- A composite number is <u>not</u> prime

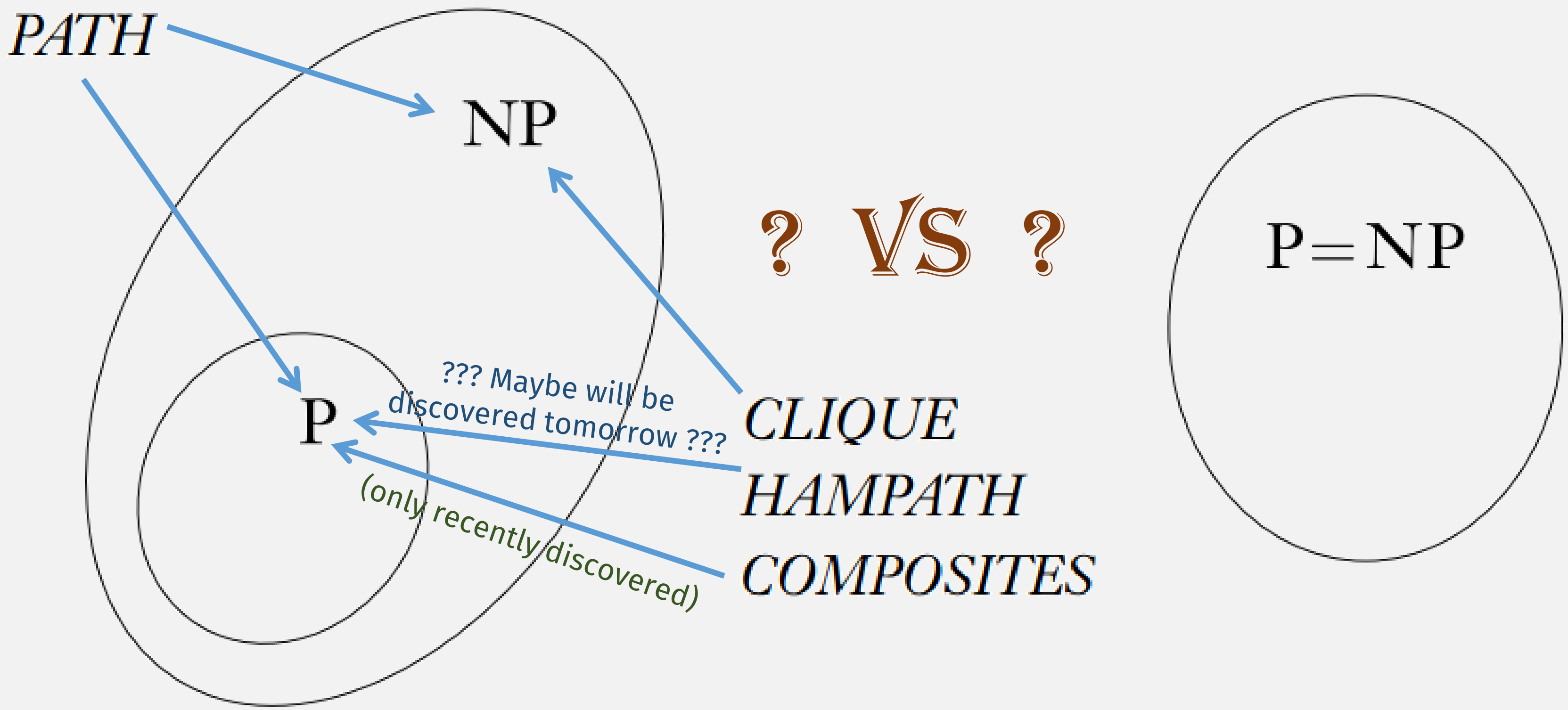
- *COMPOSITES* is polynomially verifiable
  - i.e., it's in **NP**
  - i.e., factorability is in **NP**

- A certificate could be:
  - Some factor that is not 1

- Checking existence of factors (or not, i.e., testing primality) …
  - … is also poly time
  - But only discovered <u>recently</u> (2002)!

One of the Greatest unsolved

~~HW~~ Question: Does P = NP?



How do you prove an algorithm doesn’t have a poly time algorithm?
(in general it’s hard to prove that something doesn’t exist)

Implications if **P = NP**

- Every problem with a “brute force” solution also has an efficient solution
- I.e., “unsolvable” problems are “solvable”
- BAD:
  - Cryptography needs unsolvable problems
  - Near perfect AI learning, recognition
- GOOD: Optimization problems are solved
  - Optimal resource allocation could fix all the world’s (food, energy, space …) problems?

Progress on whether **P = NP** ?

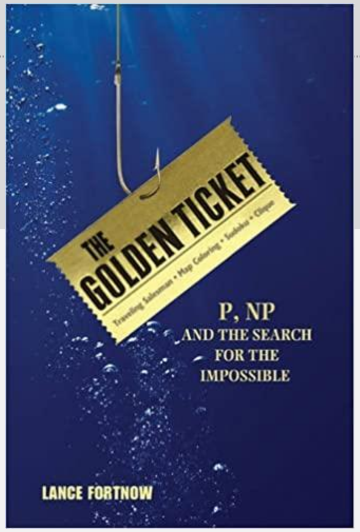
- Some, but still not close

$$P \stackrel{?}{=} NP$$

Scott Aaronson*

The Status of the P Versus NP Problem

By Lance Fortnow
Communications of the ACM, September 2009, Vol. 52 No. 9, Pages 78-86
10.1145/1562164.1562186



- One important concept discovered:
  - **NP**-Completeness

NP-Completeness

Must look at all langs, can't just look at a single lang

DEFINITION

A language B is ***NP-complete*** if it satisfies two conditions:

1. B is in NP, and easy
2. every A in NP is polynomial time reducible to B. hard????

What's this?

- How does this help the **P** = **NP** problem?

THEOREM

If B is NP-complete and $B \in \mathrm{P}$, then $\mathrm{P} = \mathrm{NP}$.

Flashback: Mapping Reducibility

Language A is ***mapping reducible*** to language B, written $\boxed{A \leq_m B}$, if there is a $\boxed{\text{computable function } f\colon \Sigma^* \longrightarrow \Sigma^*}$, where for every w,

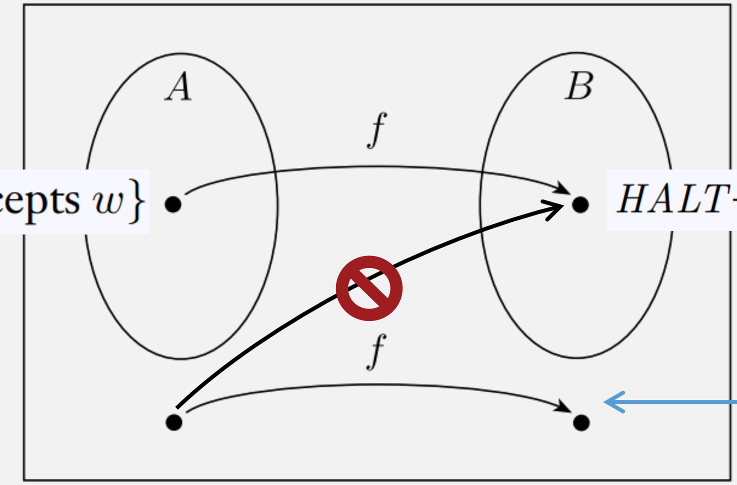
$$\boxed{w \in A \iff f(w) \in B.}$$

The function f is called the ***reduction*** from A to B.

<u>IMPORTANT</u>: "if and only if" ...

To show <u>mapping reducibility</u>:
1. create **computable fn**
2. and then show **forward direction**
3. and **reverse direction**
   (or **contrapositive of forward direction**)

$A_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ accepts } w\}$



$HALT_{TM} = \{\langle M, w\rangle \mid M \text{ is a TM and } M \text{ halts on input } w\}$

... means $\boxed{\overline{A} \leq_m \overline{B}}$

A function $f\colon \Sigma^* \longrightarrow \Sigma^*$ is a ***computable function*** if some Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Polynomial Time Mapping Reducibility

To show poly time mapping reducibility:
1. create **computable fn**
2. **show computable fn runs in poly time**
3. then show **forward direction**
4. and show **reverse direction**
   (or **contrapositive** of **reverse direction**)

Language A is ***mapping reducible*** to language B, written $A \leq_m B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \iff f(w) \in B.$$

The function f is called the ***reduction*** from A to B.

Language A is ***polynomial time mapping reducible***, or simply ***polynomial time reducible***, to language B, written $\boxed{A \leq_{\mathrm{P}} B}$, if a polynomial time computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$ exists, where for every w,

$$\boxed{w \in A \iff f(w) \in B.}$$

Don't forget: "if and only if" ...

The function f is called the ***polynomial time reduction*** of A to B.

A function $f\colon \Sigma^* \longrightarrow \Sigma^*$ is a poly time ***computable function*** if some poly time Turing machine M, on every input w, halts with just $f(w)$ on its tape.

Flashback: If $A \leq_m B$ and B is decidable, then A is decidable.

Has a decider

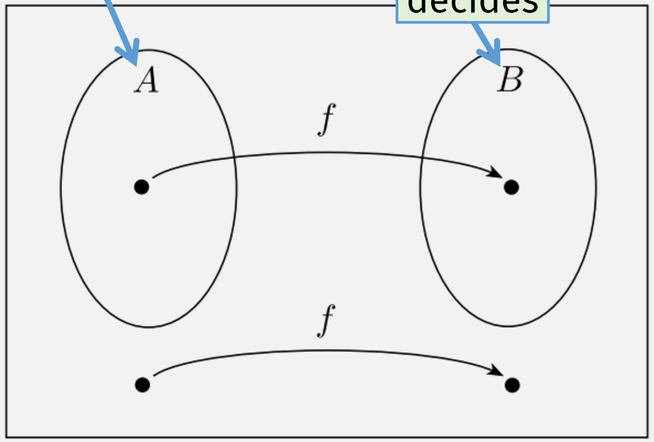
PROOF We let M be the decider for B and f be the reduction from A to B. We describe a decider N for A as follows.

N = "On input w:

1. Compute $f(w)$.
2. Run M on input $f(w)$ and output whatever M outputs."

decides

decides



This proof only works because of the if-and-only-if requirement

Language A is ***mapping reducible*** to language B, written $A \leq_m B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$\boxed{w \in A \iff f(w) \in B.}$$

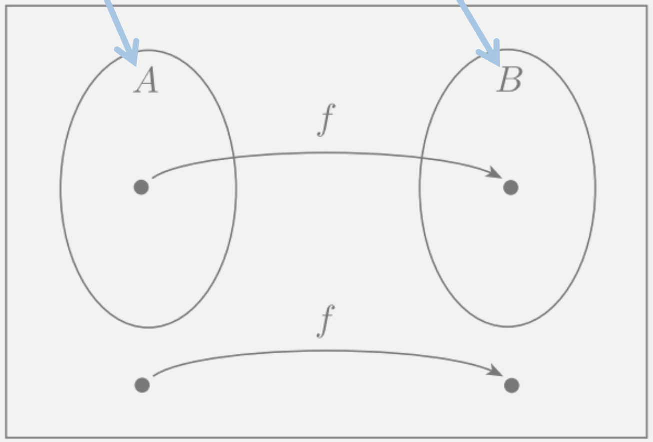
The function f is called the ***reduction*** from A to B.

Thm: If $A \leq_{\cancel{\mathrm{m}}\,\mathbf{P}} B$ and B ~~is decidable,~~ $\in \mathrm{P}$ then A ~~is decidable.~~ $\in \mathrm{P}$

PROOF We let M be the decider for B and f be the reduction from A to B. We describe a decider N for A as follows.

N = "On input w:

1. Compute $f(w)$.
2. Run M on input $f(w)$ and output whatever M outputs."



Language A is ***mapping reducible*** to language B, written $A \leq_{\mathrm{m}} B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \iff f(w) \in B.$$

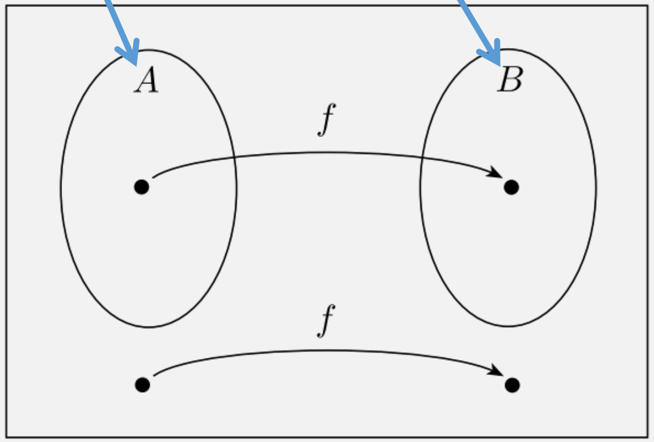
The function f is called the ***reduction*** from A to B.

Thm: If $A \leq_{\cancel{\mathrm{m}}\,\mathbf{P}} B$ and B $\in$ P ~~is decidable,~~ then A $\in$ P ~~is decidable.~~

PROOF We let M be the poly time decider for B and f be the poly time reduction from A to B. We describe a poly time decider N for A as follows.

$N =$ "On input w:

1. Compute $f(w)$.
2. Run M on input $f(w)$ and output whatever M outputs."



Language A is poly time ***mapping reducible*** to language B, written $A \leq_{\mathrm{m}} B$, if there is a computable function $f\colon \Sigma^* \longrightarrow \Sigma^*$, where for every w,

$$w \in A \iff f(w) \in B.$$

The function f is called the ***reduction*** from A to B.

THEOREM

If B is NP-complete and $B \in \mathrm{P}$, then $\mathrm{P} = \mathrm{NP}$.

To prove **P** = **NP**, must show:

1. every language in **P** is in **NP**
   - Trivially true (why?)
2. every language in **NP** is in **P**
   - Given a language $A \in$ **NP** ...
   - ... can poly time mapping reduce A to B
     - because B is **NP**-Complete
   - Then A also $\in$ **P** ...
     - Because $A \leq_{\mathrm{P}} B$ and $B \in \mathrm{P}$, then $A \in \mathrm{P}$

DEFINITION

A language B is ***NP-complete*** if it satisfies two conditions:

1. B is in NP, and
2. every A in NP is polynomial time reducible to B.

Next: **How to do poly time mapping reducibility**

Thus, **if a language** B **is NP**-complete and **in P**, **then P = NP**

Next Time: *3SAT* is polynomial time reducible to *CLIQUE*.